smaller) of the two numbers $2 - \sqrt{4-\varepsilon}$ and $\sqrt{4+\varepsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4-\varepsilon}$ and $\sqrt{4+\varepsilon}$ to make $|f(x) - 4| < \varepsilon$. For all x,

$$|f(x) - 4| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 2| < \delta.$$

This completes the proof for $\varepsilon < 4$.

If $\varepsilon \geq 4$, then we take δ to be the distance from $x = 2$ to the nearer endpoint of the interval $\left(0, \sqrt{4+\varepsilon}\right)$. In other words, take $\delta = \min\left\{2, \sqrt{4+\varepsilon} - 2\right\}$. (See Figure 2.23.) ■

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather, we appeal to general theorems about limits, in particular the theorems of Section 2.2. The definition is used to prove these theorems (Appendix 5). As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 6 Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, prove that

$$\lim_{x\to c} (f(x) + g(x)) = L + M.$$

Solution Let $\varepsilon > 0$ be given. We want to find a positive number δ such that

$$|f(x) + g(x) - (L + M)| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta.$$

Regrouping terms, we get

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M|. \end{aligned}$$

Triangle Inequality: $|a + b| \leq |a| + |b|$

Since $\lim_{x\to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon/2 \qquad \text{whenever} \qquad 0 < |x - c| < \delta_1.$$

Can find δ_1 since $\lim_{x\to c} f(x) = L$

Similarly, since $\lim_{x\to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$|g(x) - M| < \varepsilon/2 \qquad \text{whenever} \qquad 0 < |x - c| < \delta_2.$$

Can find δ_2 since $\lim_{x\to c} g(x) = M$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \varepsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \varepsilon/2$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{x\to c}(f(x) + g(x)) = L + M$. ■

EXERCISES 2.3

Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) on the x-axis with the point c inside. Then find a value of $\delta > 0$ such that $a < x < b$ whenever $0 < |x - c| < \delta$.

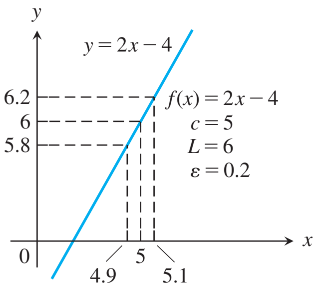
1. $a = 1, \quad b = 7, \quad c = 5$
2. $a = 1, \quad b = 7, \quad c = 2$
3. $a = -7/2, \quad b = -1/2, \quad c = -3$
4. $a = -7/2, \quad b = -1/2, \quad c = -3/2$
5. $a = 4/9, \quad b = 4/7, \quad c = 1/2$
6. $a = 2.7591, \quad b = 3.2391, \quad c = 3$

Finding Deltas Graphically

In Exercises 7–14, use the graphs to find a $\delta > 0$ such that

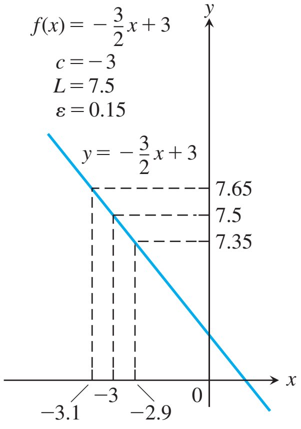
$$\left|f(x) - L\right| < \varepsilon \quad \text{whenever} \quad 0 < \left|x - c\right| < \delta.$$

7.



NOT TO SCALE

8.



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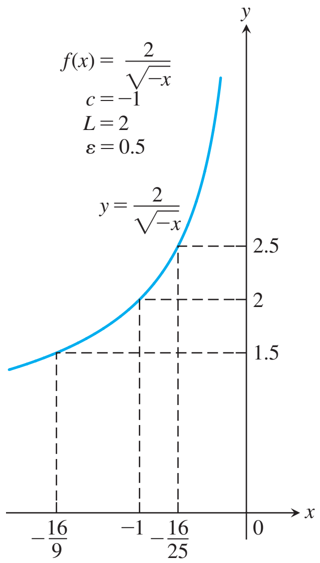

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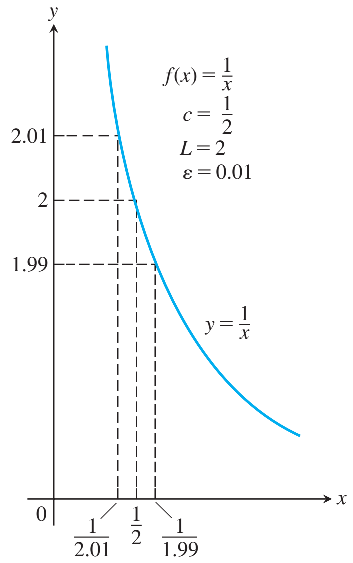

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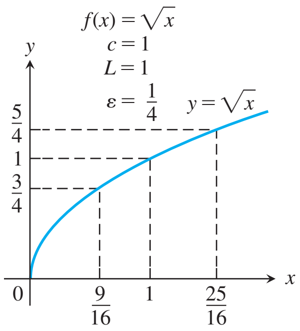

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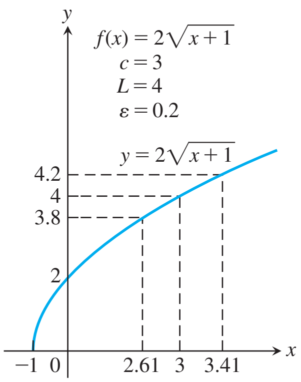

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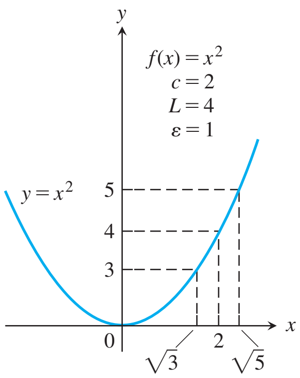

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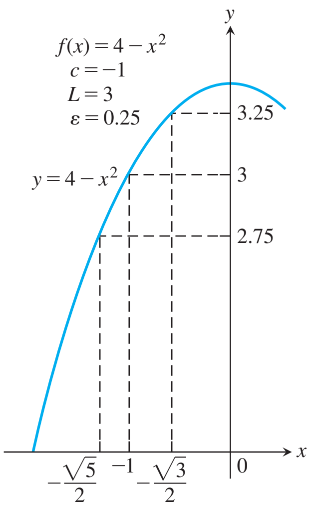

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Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, c, and $\varepsilon > 0$. In each case, find an open interval about c on which the inequality $\left|f(x) - L\right| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < \left|x - c\right| < \delta$ the inequality $\left|f(x) - L\right| < \varepsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad c = 4, \quad \varepsilon = 0.01$

16. $f(x) = 2x - 2, \quad L = -6, \quad c = -2, \quad \varepsilon = 0.02$

17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad c = 0, \quad \varepsilon = 0.1$

18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad c = 1/4, \quad \varepsilon = 0.1$

19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad c = 10, \quad \varepsilon = 1$

20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad c = 23, \quad \varepsilon = 1$

21. $f(x) = 1/x, \quad L = 1/4, \quad c = 4, \quad \varepsilon = 0.05$

22. $f(x) = x^2, \quad L = 3, \quad c = \sqrt{3}, \quad \varepsilon = 0.1$

23. $f(x) = x^2, \quad L = 4, \quad c = -2, \quad \varepsilon = 0.5$

24. $f(x) = 1/x, \quad L = -1, \quad c = -1, \quad \varepsilon = 0.1$

25. $f(x) = x^2 - 5, \quad L = 11, \quad c = 4, \quad \varepsilon = 1$

26. $f(x) = 120/x, \quad L = 5, \quad c = 24, \quad \varepsilon = 1$

27. $f(x) = mx, \quad m > 0, \quad L = 2m, \quad c = 2, \quad \varepsilon = 0.03$

28. $f(x) = mx, \quad m > 0, \quad L = 3m, \quad c = 3, \quad \varepsilon = c > 0$

29. $f(x) = mx + b, \quad m > 0, \quad L = (m/2) + b,$
$c = 1/2, \quad \varepsilon = c > 0$

30. $f(x) = mx + b, \quad m > 0, \quad L = m + b, \quad c = 1, \quad \varepsilon = 0.05$

Using the Formal Definition

Each of Exercises 31–36 gives a function $f(x)$, a point c, and a positive number ε. Find $L = \lim_{x \to c} f(x)$. Then find a number $\delta > 0$ such that

$$\left|f(x) - L\right| < \varepsilon \quad \text{whenever} \quad 0 < \left|x - c\right| < \delta.$$

31. $f(x) = 3 - 2x, \quad c = 3, \quad \varepsilon = 0.02$

32. $f(x) = -3x - 2, \quad c = -1, \quad \varepsilon = 0.03$

33. $f(x) = \dfrac{x^2 - 4}{x - 2}, \quad c = 2, \quad \varepsilon = 0.05$

34. $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}, \quad c = -5, \quad \varepsilon = 0.05$

35. $f(x) = \sqrt{1 - 5x}, \quad c = -3, \quad \varepsilon = 0.5$

36. $f(x) = 4/x, \quad c = 2, \quad \varepsilon = 0.4$

Prove the limit statements in Exercises 37–50.

37. $\lim_{x\to 4} (9 - x) = 5$

38. $\lim_{x\to 3} (3x - 7) = 2$

39. $\lim_{x\to 9} \sqrt{x - 5} = 2$

40. $\lim_{x\to 0} \sqrt{4 - x} = 2$

41. $\lim_{x\to 1} f(x) = 1 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

42. $\lim_{x\to -2} f(x) = 4 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$

43. $\lim_{x\to 1} \dfrac{1}{x} = 1$

44. $\lim_{x\to\sqrt{3}} \dfrac{1}{x^2} = \dfrac{1}{3}$

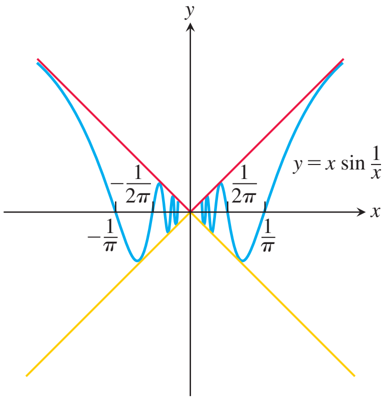
45. $\lim_{x\to -3} \dfrac{x^2 - 9}{x + 3} = -6$

46. $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1} = 2$

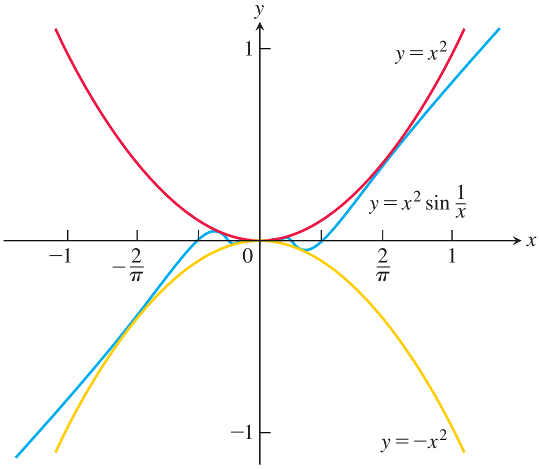
47. $\lim_{x\to 1} f(x) = 2 \quad \text{if} \quad f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$

48. $\lim_{x\to 0} f(x) = 0 \quad \text{if} \quad f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$

49. $\lim_{x\to 0} x \sin \dfrac{1}{x} = 0$



50. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x} = 0$



Theory and Examples

51. Define what it means to say that $\lim_{x\to 0} g(x) = k$.

52. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{h\to 0} f(h + c) = L$.

53. A wrong statement about limits Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches c if $f(x)$ gets closer to L as x approaches c.

Explain why the function in your example does not have the given value of L as a limit as $x \to c$.

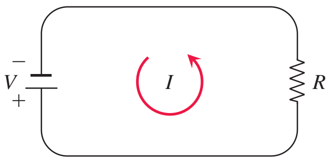
54. Another wrong statement about limits Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches c if, given any $\varepsilon > 0$, there exists a value of x for which $|f(x) - L| < \varepsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \to c$.

T **55. Grinding engine cylinders** Before contracting to grind engine cylinders to a cross-sectional area of 9 in^2, you need to know how much deviation from the ideal cylinder diameter of $c = 3.385$ in. you can allow and still have the area come within 0.01 in^2 of the required 9 in^2. To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 9| \leq 0.01$. What interval do you find?

56. Manufacturing electrical resistors Ohm's law for electrical circuits like the one shown in the accompanying figure states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the value $I_0 = 5$?



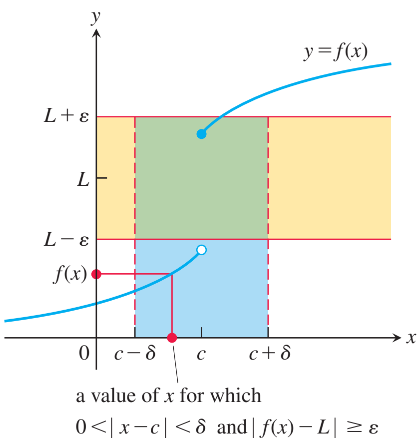
When Is a Number L Not the Limit of $f(x)$ as $x \to c$?

Showing L is not a limit We can prove that $\lim_{x\to c} f(x) \neq L$ by providing an $\varepsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

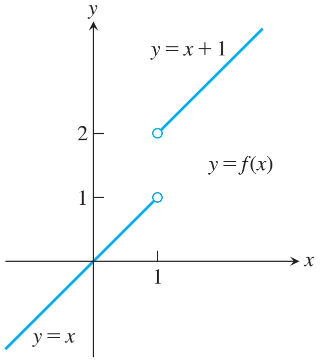
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

We accomplish this for our candidate ε by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - c| < \delta \qquad \text{and} \qquad |f(x) - L| \geq \varepsilon.$$



57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1. \end{cases}$



a. Let $\varepsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

$$|f(x) - 2| < 1/2 \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

That is, for each $\delta > 0$ show that there is a value of x such that

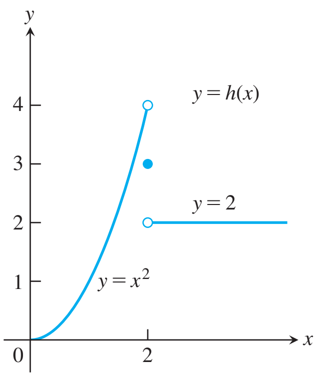
$$0 < |x - 1| < \delta \qquad \text{and} \qquad |f(x) - 2| \geq 1/2.$$

This will show that $\lim_{x \to 1} f(x) \neq 2$.

b. Show that $\lim_{x \to 1} f(x) \neq 1$.

c. Show that $\lim_{x \to 1} f(x) \neq 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$



Show that

a. $\lim_{x \to 2} h(x) \neq 4$

b. $\lim_{x \to 2} h(x) \neq 3$

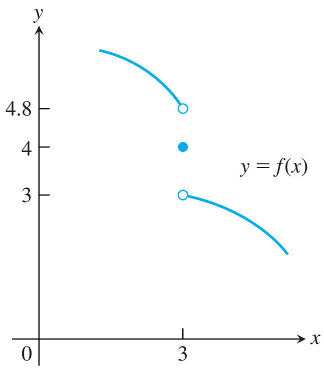
c. $\lim_{x \to 2} h(x) \neq 2$

59. For the function graphed here, explain why

a. $\lim_{x \to 3} f(x) \neq 4$

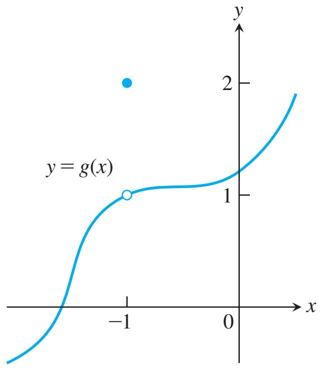
b. $\lim_{x \to 3} f(x) \neq 4.8$

c. $\lim_{x \to 3} f(x) \neq 3$



60. a. For the function graphed here, show that $\lim_{x \to -1} g(x) \neq 2$.

b. Does $\lim_{x \to -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



COMPUTER EXPLORATIONS

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

a. Plot the function $y = f(x)$ near the point c being approached.

b. Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.

c. Using the value $\varepsilon = 0.2$, graph the banding lines $y_1 = L - \varepsilon$ and $y_2 = L + \varepsilon$ together with the function f near c.

d. From your graph in part (c), estimate a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Test your estimate by plotting f, y_1, and y_2 over the interval $0 < |x - c| < \delta$. For your viewing window use $c - 2\delta \leq x \leq c + 2\delta$ and $L - 2\varepsilon \leq y \leq L + 2\varepsilon$. If any function values lie outside the interval $[L - \varepsilon, L + \varepsilon]$, your choice of δ was too large. Try again with a smaller estimate.

e. Repeat parts (c) and (d) successively for $\varepsilon = 0.1, 0.05$, and 0.001.

61. $f(x) = \dfrac{x^4 - 81}{x - 3}, \quad c = 3$

62. $f(x) = \dfrac{5x^3 + 9x^2}{2x^5 + 3x^2}, \quad c = 0$

63. $f(x) = \dfrac{\sin 2x}{3x}, \quad c = 0$

64. $f(x) = \dfrac{x(1 - \cos x)}{x - \sin x}, \quad c = 0$

65. $f(x) = \dfrac{\sqrt[3]{x} - 1}{x - 1}, \quad c = 1$

66. $f(x) = \dfrac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}, \quad c = 1$