EXAMPLE 6 Find $\lim_{t\to 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t\to 0} \frac{1}{3}\cdot\frac{1}{t}\cdot\frac{\sin t}{\cos t}\cdot\frac{1}{\cos 2t} \\
&= \frac{1}{3}\lim_{t\to 0} \frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t} \\
&= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.
\end{aligned}$$

Eq. (1) and Example 11b in Section 2.2

EXAMPLE 7 Show that for nonzero constants A and B.

$$\lim_{\theta\to 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

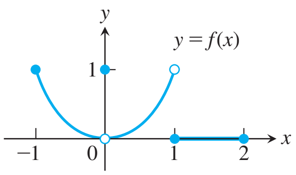
Solution

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin A\theta}{\sin B\theta} &= \lim_{\theta\to 0} \frac{\sin A\theta}{A\theta} A\theta \frac{B\theta}{\sin B\theta}\frac{1}{B\theta} && \text{Multiply and divide by } A\theta \text{ and } B\theta. \\
&= \lim_{\theta\to 0} \frac{\sin A\theta}{A\theta}\,\frac{B\theta}{\sin B\theta}\,\frac{A}{B} && \lim_{u\to 0}\frac{\sin u}{u} = 1,\ \text{with } u = A\theta \\
&= \lim_{\theta\to 0} (1)(1)\frac{A}{B} && \lim_{v\to 0}\frac{v}{\sin v} = 1,\ \text{with } v = B\theta \\
&= \frac{A}{B}.
\end{aligned}$$

EXERCISES 2.4

Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$

b. $\lim_{x\to 0^-} f(x) = 0$

c. $\lim_{x\to 0^-} f(x) = 1$

d. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

e. $\lim_{x\to 0} f(x)$ exists.

f. $\lim_{x\to 0} f(x) = 0$

g. $\lim_{x\to 0} f(x) = 1$

h. $\lim_{x\to 1} f(x) = 1$

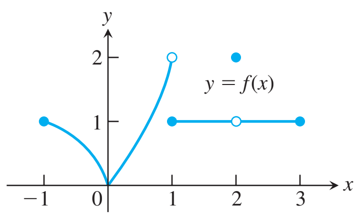
i. $\lim_{x\to 1} f(x) = 0$

j. $\lim_{x\to 2^-} f(x) = 2$

k. $\lim_{x\to -1^-} f(x)$ does not exist.

l. $\lim_{x\to 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$

b. $\lim_{x\to 2} f(x)$ does not exist.

c. $\lim_{x\to 2} f(x) = 2$

d. $\lim_{x\to 1^-} f(x) = 2$

e. $\lim_{x\to 1^+} f(x) = 1$

f. $\lim_{x\to 1} f(x)$ does not exist.

g. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

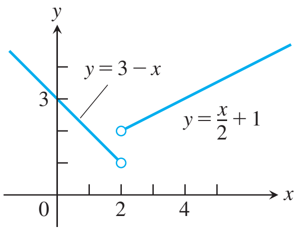
h. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j. $\lim_{x\to -1^-} f(x) = 0$

k. $\lim_{x\to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$



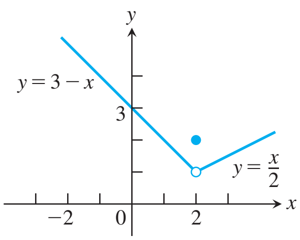
a. Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$.

b. Does $\lim_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.

d. Does $\lim_{x \to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



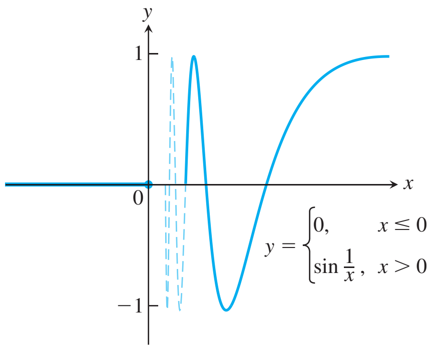
a. Find $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$.

d. Does $\lim_{x \to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin \dfrac{1}{x}, & x > 0. \end{cases}$

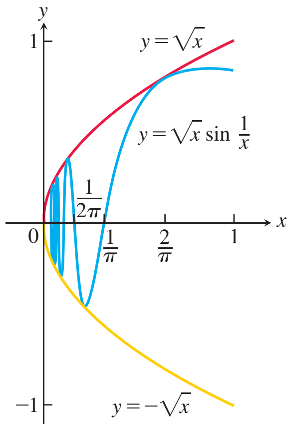


a. Does $\lim_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x \to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x \to 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x} \sin(1/x)$.



a. Does $\lim_{x \to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x \to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x \to 0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \ne 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x \to c} f(x)$ exist?

c. At what points does the left-hand limit exist but not the right-hand limit?

d. At what points does the right-hand limit exist but not the left-hand limit?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–20.

11. $\lim\limits_{x \to -0.5^-} \sqrt{\dfrac{x + 2}{x + 1}}$

12. $\lim\limits_{x \to 1^+} \sqrt{\dfrac{x - 1}{x + 2}}$

13. $\lim\limits_{x \to -2^+} \left(\dfrac{x}{x + 1}\right)\left(\dfrac{2x + 5}{x^2 + x}\right)$

14. $\lim_{x \to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

15. $\lim_{h \to 0^+} \frac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$

16. $\lim_{h \to 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$

17. a. $\lim_{x \to -2^+} (x + 3)\frac{|x + 2|}{x + 2}$ **b.** $\lim_{x \to -2^-} (x + 3)\frac{|x + 2|}{x + 2}$

18. a. $\lim_{x \to 1^+} \frac{\sqrt{2x}(x - 1)}{|x - 1|}$ **b.** $\lim_{x \to 1^-} \frac{\sqrt{2x}(x - 1)}{|x - 1|}$

19. a. $\lim_{x \to 0^+} \frac{|\sin x|}{\sin x}$ **b.** $\lim_{x \to 0^-} \frac{|\sin x|}{\sin x}$

20. a. $\lim_{x \to 0^+} \frac{1 - \cos x}{|\cos x - 1|}$ **b.** $\lim_{x \to 0^-} \frac{\cos x - 1}{|\cos x - 1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 21 and 22.

21. a. $\lim_{\theta \to 3^+} \frac{\lfloor \theta \rfloor}{\theta}$ **b.** $\lim_{\theta \to 3^-} \frac{\lfloor \theta \rfloor}{\theta}$

22. a. $\lim_{t \to 4^+} (t - \lfloor t \rfloor)$ **b.** $\lim_{t \to 4^-} (t - \lfloor t \rfloor)$

Using $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Find the limits in Exercises 23–46.

23. $\lim_{\theta \to 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

24. $\lim_{t \to 0} \frac{\sin kt}{t}$ (k constant)

25. $\lim_{y \to 0} \frac{\sin 3y}{4y}$

26. $\lim_{h \to 0^-} \frac{h}{\sin 3h}$

27. $\lim_{x \to 0} \frac{\tan 2x}{x}$

28. $\lim_{t \to 0} \frac{2t}{\tan t}$

29. $\lim_{x \to 0} \frac{x \csc 2x}{\cos 5x}$

30. $\lim_{x \to 0} 6x^2(\cot x)(\csc 2x)$

31. $\lim_{x \to 0} \frac{x + x \cos x}{\sin x \cos x}$

32. $\lim_{x \to 0} \frac{x^2 - x + \sin x}{2x}$

33. $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\sin 2\theta}$

34. $\lim_{x \to 0} \frac{x - x \cos x}{\sin^2 3x}$

35. $\lim_{t \to 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

36. $\lim_{h \to 0} \frac{\sin(\sin h)}{\sin h}$

37. $\lim_{\theta \to 0} \frac{\sin \theta}{\sin 2\theta}$

38. $\lim_{x \to 0} \frac{\sin 5x}{\sin 4x}$

39. $\lim_{\theta \to 0} \theta \cos \theta$

40. $\lim_{\theta \to 0} \sin \theta \cot 2\theta$

41. $\lim_{x \to 0} \frac{\tan 3x}{\sin 8x}$

42. $\lim_{y \to 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

43. $\lim_{\theta \to 0} \frac{\tan \theta}{\theta^2 \cot 3\theta}$

44. $\lim_{\theta \to 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

45. $\lim_{x \to 0} \frac{1 - \cos 3x}{2x}$

46. $\lim_{x \to 0} \frac{\cos^2 x - \cos x}{x^2}$

Theory and Examples

47. Once you know $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x \to a} f(x)$? Give reasons for your answer.

48. If you know that $\lim_{x \to c} f(x)$ exists, can you find its value by calculating $\lim_{x \to c^+} f(x)$? Give reasons for your answer.

49. Suppose that f is an odd function of x. Does knowing that $\lim_{x \to 0^+} f(x) = 3$ tell you anything about $\lim_{x \to 0^-} f(x)$? Give reasons for your answer.

50. Suppose that f is an even function of x. Does knowing that $\lim_{x \to 2^-} f(x) = 7$ tell you anything about either $\lim_{x \to -2^-} f(x)$ or $\lim_{x \to -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

51. Given $\varepsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x - 5} < \varepsilon$. What limit is being verified and what is its value?

52. Given $\varepsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4 - x} < \varepsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 53 and 54.

53. $\lim_{x \to 0^-} \frac{x}{|x|} = -1$

54. $\lim_{x \to 2^+} \frac{x - 2}{|x - 2|} = 1$

55. Greatest integer function Find **(a)** $\lim_{x \to 400^+} \lfloor x \rfloor$ and **(b)** $\lim_{x \to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x \to 400} \lfloor x \rfloor$? Give reasons for your answer.

56. One-sided limits Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find **(a)** $\lim_{x \to 0^+} f(x)$ and **(b)** $\lim_{x \to 0^-} f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x \to 0} f(x)$? Give reasons for your answer.

2.5 Continuity

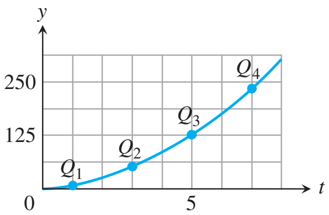


FIGURE 2.34 Connecting plotted points.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.34). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.