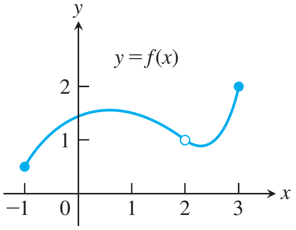
EXERCISES 2.5

Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

1.



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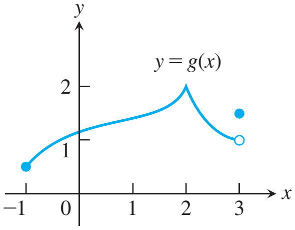

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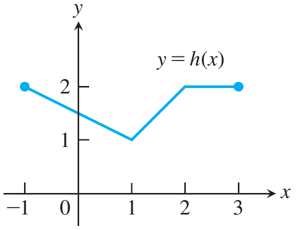

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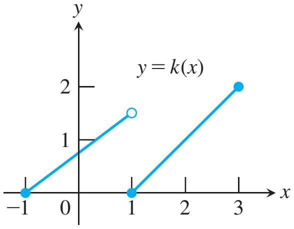
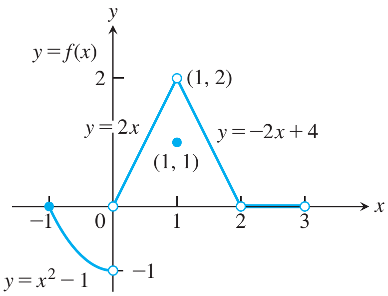


Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. a. Does $f(-1)$ exist?

b. Does $\lim_{x\to -1^+} f(x)$ exist?

c. Does $\lim_{x\to -1^+} f(x) = f(-1)$?

d. Is f continuous at $x = -1$?

6. a. Does $f(1)$ exist?

b. Does $\lim_{x\to 1} f(x)$ exist?

c. Does $\lim_{x\to 1} f(x) = f(1)$?

d. Is f continuous at $x = 1$?

7. a. Is f defined at $x = 2$? (Look at the definition of f.)

b. Is f continuous at $x = 2$?

8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

11. Exercise 1, Section 2.4

12. Exercise 2, Section 2.4

At what points are the functions in Exercises 13–32 continuous?

13. $y = \dfrac{1}{x-2} - 3x$

14. $y = \dfrac{1}{(x+2)^2} + 4$

15. $y = \dfrac{x+1}{x^2 - 4x + 3}$

16. $y = \dfrac{x+3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$

18. $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$

19. $y = \dfrac{\cos x}{x}$

20. $y = \dfrac{x+2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \dfrac{\pi x}{2}$

23. $y = \dfrac{x \tan x}{x^2 + 1}$

24. $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

25. $y = \sqrt{2x + 3}$

26. $y = \sqrt[4]{3x - 1}$

27. $y = (2x - 1)^{1/3}$

28. $y = (2 - x)^{1/5}$

29. $g(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x \ne 3 \\ 5, & x = 3 \end{cases}$

30. $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \ne 2, x \ne -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$

31. $f(x) = \begin{cases} 1 - x, & x < 0 \\ e^x, & 0 \le x \le 1 \\ x^2 + 2, & x > 1 \end{cases}$

32. $f(x) = \dfrac{x+3}{2 - e^x}$

Limits Involving Trigonometric Functions

Find the limits in Exercises 33–40. Are the functions continuous at the point being approached?

33. $\lim_{x\to\pi} \sin(x - \sin x)$

34. $\lim_{t\to 0} \sin\left(\dfrac{\pi}{2}\cos(\tan t)\right)$

35. $\lim_{y\to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

36. $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos(\sin x^{1/3})\right)$

37. $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right)$

38. $\lim_{x\to \pi/6} \sqrt{\csc^2 x + 5\sqrt{3}\tan x}$

39. $\lim_{x\to 0^+} \sin\left(\frac{\pi}{2}e^{\sqrt{x}}\right)$

40. $\lim_{x\to 1} \cos^{-1}\left(\ln\sqrt{x}\right)$

Continuous Extensions

41. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

42. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

43. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

44. Define $g(4)$ in a way that extends

$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$

to be continuous at $x = 4$.

45. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \ge 3 \end{cases}$$

continuous at every x?

46. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \ge -2 \end{cases}$$

continuous at every x?

47. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \ge 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x?

48. For what value of b is

$$g(x) = \begin{cases} \dfrac{x - b}{b + 1}, & x < 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every x?

49. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \le -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \ge 1 \end{cases}$$

continuous at every x?

50. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \le 0 \\ x^2 + 3a - b, & 0 < x \le 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?

T In Exercises 51–54, graph the function f to see whether it appears to have a continuous extension to the origin. If it does, use Trace and Zoom to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at the origin from the right or from the left? If so, what do you think the extended function's value(s) should be?

51. $f(x) = \dfrac{10^x - 1}{x}$

52. $f(x) = \dfrac{10^{|x|} - 1}{x}$

53. $f(x) = \dfrac{\sin x}{|x|}$

54. $f(x) = (1 + 2x)^{1/x}$

Theory and Examples

55. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

56. Explain why the equation $\cos x = x$ has at least one solution.

57. **Roots of a cubic** Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

58. **A function value** Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x.

59. **Solving an equation** If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals **(a)** π; **(b)** $-\sqrt{3}$; **(c)** 5,000,000.

60. Explain why the following five statements ask for the same information.

   a. Find the roots of $f(x) = x^3 - 3x - 1$.

   b. Find the x-coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

   c. Find all the values of x for which $x^3 - 3x = 1$.

   d. Find the x-coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

   e. Solve the equation $x^3 - 3x - 1 = 0$.

61. **Removable discontinuity** Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

62. **Nonremovable discontinuity** Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

63. **A function discontinuous at every point**

   a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

   is discontinuous at every point.

   b. Is f right-continuous or left-continuous at any point?

64. If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

65. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

66. Discontinuous composite of continuous functions Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 9? Give reasons for your answer.

67. Never-zero continuous functions Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

68. Stretching a rubber band Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

69. A fixed point theorem Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

70. The sign-preserving property of continuous functions Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$.

71. Prove that f is continuous at c if and only if

$$\lim_{h \to 0} f(c + h) = f(c).$$

72. Use Exercise 71 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$
$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 73–80 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

73. $x^3 - 3x - 1 = 0$

74. $2x^3 - 2x^2 - 2x + 1 = 0$

75. $x(x - 1)^2 = 1$ (one root)

76. $x^x = 2$

77. $\sqrt{x} + \sqrt{1 + x} = 4$

78. $x^3 - 15x + 1 = 0$ (three roots)

79. $\cos x = x$ (one root). Make sure you are using radian mode.

80. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

2.6 Limits Involving Infinity; Asymptotes of Graphs

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to *infinite limits*. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

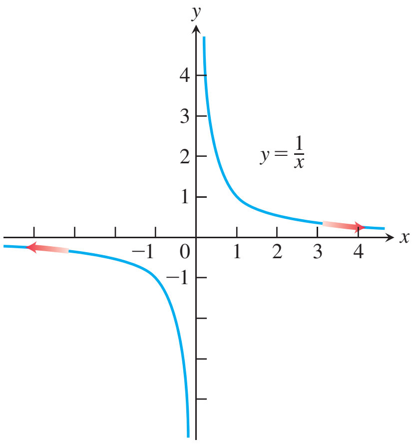


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.49). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \infty$ or $x \to -\infty$, or that 0 is a *limit of $f(x) = 1/x$ at infinity and at negative infinity*. Here are precise definitions for the limit of a function whose domain contains positive or negative numbers of unbounded magnitude.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

2. We say that $f(x)$ has the **limit L as x approaches negative infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$