as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2} + 1\right) + \left(\frac{1}{2x - 4}\right).$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{For } |x| \text{ large, } \frac{1}{2x-4} \text{ is near 0.}$$

$$f(x) \approx \frac{1}{2x - 4} \qquad \text{For } x \text{ near 2, this term is very large in absolute value.}$$

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when $|x|$ is large and the contribution of $1/(2x - 4)$ to the total value of f is insignificant. It behaves like $1/(2x - 4)$ when x is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x approaches ∞ or $-\infty$, and we say that $1/(2x - 4)$ dominates when x approaches 2. **Dominant terms** like these help us predict a function's behavior.

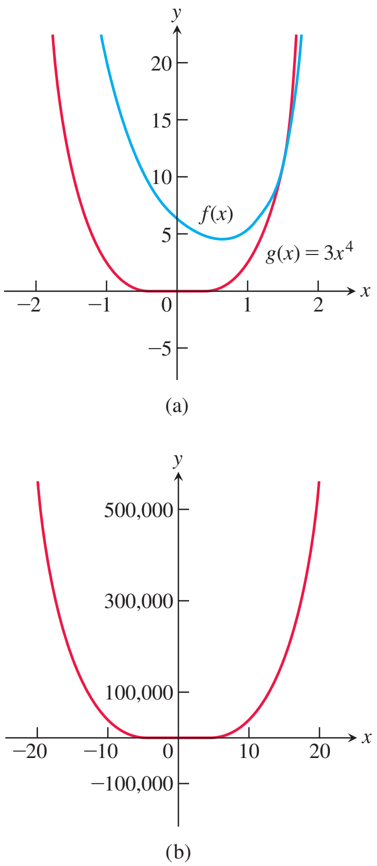


FIGURE 2.69 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 20).

EXAMPLE 20 Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x, they behave similarly for $|x|$ very large, in the sense that their ratios approach 1 as $x \to \infty$ or $x \to -\infty$.

Solution The graphs of f and g behave quite differently near the origin (Figure 2.69a), but appear as virtually identical on a larger scale (Figure 2.69b).

We can test that the term $3x^4$ in f, represented graphically by g, dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned} \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\ &= 1, \end{aligned}$$

which means that f and g appear nearly identical when $|x|$ is large. ■

EXERCISES 2.6

Finding Limits

1. For the function f whose graph is given, determine the following limits.

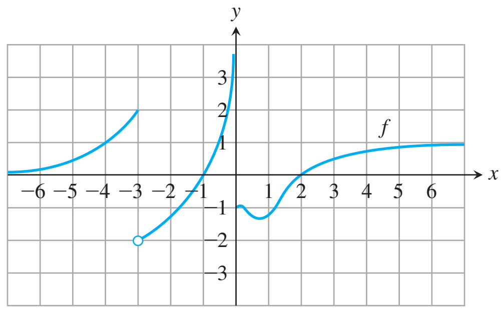
a. $\lim_{x\to 2} f(x)$ **b.** $\lim_{x\to -3^+} f(x)$

c. $\lim_{x\to -3^-} f(x)$ **d.** $\lim_{x\to -3} f(x)$

e. $\lim_{x\to 0^+} f(x)$ **f.** $\lim_{x\to 0^-} f(x)$

g. $\lim_{x\to 0} f(x)$ **h.** $\lim_{x\to \infty} f(x)$

i. $\lim_{x\to -\infty} f(x)$



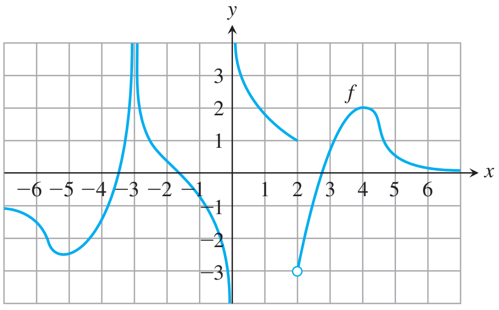
2. For the function f whose graph is given, determine the following limits.

a. $\lim_{x\to 4} f(x)$ b. $\lim_{x\to 2^+} f(x)$ c. $\lim_{x\to 2^-} f(x)$

d. $\lim_{x\to 2} f(x)$ e. $\lim_{x\to -3^+} f(x)$ f. $\lim_{x\to -3^-} f(x)$

g. $\lim_{x\to -3} f(x)$ h. $\lim_{x\to 0^+} f(x)$ i. $\lim_{x\to 0^-} f(x)$

j. $\lim_{x\to 0} f(x)$ k. $\lim_{x\to \infty} f(x)$ l. $\lim_{x\to -\infty} f(x)$



In Exercises 3–8, find the limit of each function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x) = \dfrac{2}{x} - 3$

4. $f(x) = \pi - \dfrac{2}{x^2}$

5. $g(x) = \dfrac{1}{2 + (1/x)}$

6. $g(x) = \dfrac{1}{8 - (5/x^2)}$

7. $h(x) = \dfrac{-5 + (7/x)}{3 - (1/x^2)}$

8. $h(x) = \dfrac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\lim_{x\to\infty} \dfrac{\sin 2x}{x}$

10. $\lim_{\theta\to -\infty} \dfrac{\cos\theta}{3\theta}$

11. $\lim_{t\to -\infty} \dfrac{2 - t + \sin t}{t + \cos t}$

12. $\lim_{r\to\infty} \dfrac{r + \sin r}{2r + 7 - 5\sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

13. $f(x) = \dfrac{2x + 3}{5x + 7}$

14. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

15. $f(x) = \dfrac{x + 1}{x^2 + 3}$

16. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

17. $h(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

18. $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

19. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

20. $g(x) = \dfrac{x^3 + 7x^2 - 2}{x^2 - x + 1}$

21. $f(x) = \dfrac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$

22. $h(x) = \dfrac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$

Limits as $x \to \infty$ or $x \to -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\lim_{x\to\infty} \sqrt{\dfrac{8x^2 - 3}{2x^2 + x}}$

24. $\lim_{x\to -\infty} \left(\dfrac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

25. $\lim_{x\to -\infty} \left(\dfrac{1 - x^3}{x^2 + 7x}\right)^5$

26. $\lim_{x\to\infty} \sqrt{\dfrac{x^2 - 5x}{x^3 + x - 2}}$

27. $\lim_{x\to\infty} \dfrac{2\sqrt{x} + x^{-1}}{3x - 7}$

28. $\lim_{x\to\infty} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}}$

29. $\lim_{x\to -\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

30. $\lim_{x\to\infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

31. $\lim_{x\to\infty} \dfrac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

32. $\lim_{x\to -\infty} \dfrac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

33. $\lim_{x\to\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

34. $\lim_{x\to -\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

35. $\lim_{x\to\infty} \dfrac{x - 3}{\sqrt{4x^2 + 25}}$

36. $\lim_{x\to -\infty} \dfrac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Infinite Limits

Find the limits in Exercises 37–48.

37. $\lim_{x\to 0^+} \dfrac{1}{3x}$

38. $\lim_{x\to 0^-} \dfrac{5}{2x}$

39. $\lim_{x\to 2^-} \dfrac{3}{x - 2}$

40. $\lim_{x\to 3^+} \dfrac{1}{x - 3}$

41. $\lim_{x\to -8^+} \dfrac{2x}{x + 8}$

42. $\lim_{x\to -5^-} \dfrac{3x}{2x + 10}$

43. $\lim_{x\to 7} \dfrac{4}{(x - 7)^2}$

44. $\lim_{x\to 0} \dfrac{-1}{x^2(x + 1)}$

45. **a.** $\lim_{x\to 0^+} \dfrac{2}{3x^{1/3}}$ **b.** $\lim_{x\to 0^-} \dfrac{2}{3x^{1/3}}$

46. **a.** $\lim_{x\to 0^+} \dfrac{2}{x^{1/5}}$ **b.** $\lim_{x\to 0^-} \dfrac{2}{x^{1/5}}$

47. $\lim_{x\to 0} \dfrac{4}{x^{2/5}}$

48. $\lim_{x\to 0} \dfrac{1}{x^{2/3}}$

Find the limits in Exercises 49–52.

49. $\lim_{x\to(\pi/2)^-} \tan x$

50. $\lim_{x\to(-\pi/2)^+} \sec x$

51. $\lim_{\theta\to 0^-} (1 + \csc\theta)$

52. $\lim_{\theta\to 0} (2 - \cot\theta)$

Find the limits in Exercises 53–58.

53. $\lim \dfrac{1}{x^2 - 4}$ as

a. $x \to 2^+$ **b.** $x \to 2^-$

c. $x \to -2^+$ **d.** $x \to -2^-$

54. $\lim \dfrac{x}{x^2 - 1}$ as

a. $x \to 1^+$ **b.** $x \to 1^-$

c. $x \to -1^+$ **d.** $x \to -1^-$

55. $\lim \left(\frac{x^2}{2} - \frac{1}{x}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to \sqrt[3]{2}$ **d.** $x \to -1$

56. $\lim \frac{x^2 - 1}{2x + 4}$ as

a. $x \to -2^+$ **b.** $x \to -2^-$

c. $x \to 1^+$ **d.** $x \to 0^-$

57. $\lim \frac{x^2 - 3x + 2}{x^3 - 2x^2}$ as

a. $x \to 0^+$ **b.** $x \to 2^+$

c. $x \to 2^-$ **d.** $x \to 2$

e. What, if anything, can be said about the limit as $x \to 0$?

58. $\lim \frac{x^2 - 3x + 2}{x^3 - 4x}$ as

a. $x \to 2^+$ **b.** $x \to -2^+$

c. $x \to 0^-$ **d.** $x \to 1^+$

e. What, if anything, can be said about the limit as $x \to 0$?

Find the limits in Exercises 59–62.

59. $\lim \left(2 - \frac{3}{t^{1/3}}\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

60. $\lim \left(\frac{1}{t^{3/5}} + 7\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

61. $\lim \left(\frac{1}{x^{2/3}} + \frac{2}{(x - 1)^{2/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to 1^+$ **d.** $x \to 1^-$

62. $\lim \left(\frac{1}{x^{1/3}} - \frac{1}{(x - 1)^{4/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to 1^+$ **d.** $x \to 1^-$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

63. $y = \frac{1}{x - 1}$ **64.** $y = \frac{1}{x + 1}$

65. $y = \frac{1}{2x + 4}$ **66.** $y = \frac{-3}{x - 3}$

67. $y = \frac{x + 3}{x + 2}$ **68.** $y = \frac{2x}{x + 1}$

Domains, Ranges, and Asymptotes

Determine the domain and range of each function. Use various limits to find the asymptotes and the ranges.

69. $y = 4 + \frac{3x^2}{x^2 + 1}$ **70.** $y = \frac{2x}{x^2 - 1}$

71. $y = \frac{8 - e^x}{2 + e^x}$ **72.** $y = \frac{4e^x + e^{2x}}{e^x + e^{2x}}$

73. $y = \frac{\sqrt{x^2 + 4}}{x}$ **74.** $y = \frac{x^3}{x^3 - 8}$

Inventing Graphs and Functions

In Exercises 75–78, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

75. $f(0) = 0, f(1) = 2, f(-1) = -2, \lim_{x \to -\infty} f(x) = -1$, and $\lim_{x \to \infty} f(x) = 1$

76. $f(0) = 0, \lim_{x \to \pm\infty} f(x) = 0, \lim_{x \to 0^+} f(x) = 2$, and $\lim_{x \to 0^-} f(x) = -2$

77. $f(0) = 0, \lim_{x \to \pm\infty} f(x) = 0, \lim_{x \to 1^-} f(x) = \lim_{x \to -1^+} f(x) = \infty$, $\lim_{x \to 1^+} f(x) = -\infty$, and $\lim_{x \to -1^-} f(x) = -\infty$

78. $f(2) = 1, f(-1) = 0, \lim_{x \to \infty} f(x) = 0, \lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$, and $\lim_{x \to -\infty} f(x) = 1$

In Exercises 79–82, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

79. $\lim_{x \to \pm\infty} f(x) = 0, \lim_{x \to 2^-} f(x) = \infty$, and $\lim_{x \to 2^+} f(x) = \infty$

80. $\lim_{x \to \pm\infty} g(x) = 0, \lim_{x \to 3^-} g(x) = -\infty$, and $\lim_{x \to 3^+} g(x) = \infty$

81. $\lim_{x \to -\infty} h(x) = -1, \lim_{x \to \infty} h(x) = 1, \lim_{x \to 0^-} h(x) = -1$, and $\lim_{x \to 0^+} h(x) = 1$

82. $\lim_{x \to \pm\infty} k(x) = 1, \lim_{x \to 1^-} k(x) = \infty$, and $\lim_{x \to 1^+} k(x) = -\infty$

83. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x \to \infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x \to -\infty} (f(x)/g(x))$? Give reasons for your answer.

84. Suppose that $f(x)$ and $g(x)$ are polynomials in x. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

85. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences When $x \to \pm\infty$

Find the limits in Exercises 86–92. (*Hint*: Try multiplying and dividing by the conjugate.)

86. $\lim_{x \to \infty} (\sqrt{x + 9} - \sqrt{x + 4})$

87. $\lim_{x \to \infty} (\sqrt{x^2 + 25} - \sqrt{x^2 - 1})$

88. $\lim_{x \to -\infty} (\sqrt{x^2 + 3} + x)$

89. $\lim_{x \to -\infty} (2x + \sqrt{4x^2 + 3x - 2})$

90. $\lim_{x \to \infty} (\sqrt{9x^2 - x} - 3x)$

91. $\lim_{x\to\infty}(\sqrt{x^2+3x}-\sqrt{x^2-2x})$

92. $\lim_{x\to\infty}(\sqrt{x^2+x}-\sqrt{x^2-x})$

Using the Formal Definitions

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 93 and 94.

93. If f has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

94. If f has the constant value $f(x) = k$, then $\lim_{x\to-\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 95–98.

95. $\lim_{x\to 0}\frac{-1}{x^2} = -\infty$

96. $\lim_{x\to 0}\frac{1}{|x|} = \infty$

97. $\lim_{x\to 3}\frac{-2}{(x-3)^2} = -\infty$

98. $\lim_{x\to -5}\frac{1}{(x+5)^2} = \infty$

99. Here is the definition of **infinite right-hand limit**.

> Suppose that an interval (c, d) lies in the domain of f. We say that $f(x)$ approaches infinity as x approaches c from the right, and write
>
> $$\lim_{x\to c^+} f(x) = \infty,$$
>
> if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that
>
> $$f(x) > B \quad \text{whenever} \quad c < x < c + \delta.$$

Modify the definition to cover the following cases.

a. $\lim_{x\to c^-} f(x) = \infty$

b. $\lim_{x\to c^+} f(x) = -\infty$

c. $\lim_{x\to c^-} f(x) = -\infty$

Use the formal definitions from Exercise 99 to prove the limit statements in Exercises 100–104.

100. $\lim_{x\to 0^+}\frac{1}{x} = \infty$

101. $\lim_{x\to 0^-}\frac{1}{x} = -\infty$

102. $\lim_{x\to 2^-}\frac{1}{x-2} = -\infty$

103. $\lim_{x\to 2^+}\frac{1}{x-2} = \infty$

104. $\lim_{x\to 1^-}\frac{1}{1-x^2} = \infty$

Oblique Asymptotes

Graph the rational functions in Exercises 105–110. Include the graphs and equations of the asymptotes.

105. $y = \frac{x^2}{x-1}$

106. $y = \frac{x^2+1}{x-1}$

107. $y = \frac{x^2-4}{x-1}$

108. $y = \frac{x^2-1}{2x+4}$

109. $y = \frac{x^2-1}{x}$

110. $y = \frac{x^3+1}{x^2}$

Additional Graphing Exercises

T Graph the curves in Exercises 111–114. Explain the relationship between the curve's formula and what you see.

111. $y = \frac{x}{\sqrt{4-x^2}}$

112. $y = \frac{-1}{\sqrt{4-x^2}}$

113. $y = x^{2/3} + \frac{1}{x^{1/3}}$

114. $y = \sin\left(\frac{\pi}{x^2+1}\right)$

T Graph the functions in Exercises 115 and 116. Then answer the following questions.

a. How does the graph behave as $x \to 0^+$?

b. How does the graph behave as $x \to \pm\infty$?

c. How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

115. $y = \frac{3}{2}\left(x - \frac{1}{x}\right)^{2/3}$

116. $y = \frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3}$

CHAPTER 2 Questions to Guide Your Review

1. What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?
2. What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?
3. Give an informal or intuitive definition of the limit

   $$\lim_{x\to c} f(x) = L.$$

   Why is the definition "informal"? Give examples.
4. Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain and give examples.
5. What function behaviors might occur for which the limit may fail to exist? Give examples.
6. What theorems are available for calculating limits? Give examples of how the theorems are used.
7. How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.
8. What is the value of $\lim_{\theta\to 0}((\sin\theta)/\theta)$? Does it matter whether θ is measured in degrees or radians? Explain.