we interpret the difference quotient as an average rate of change (Section 2.1), then the derivative gives the function's instantaneous rate of change with respect to x at the point $x = x_0$. We study this interpretation in Section 3.4.

EXAMPLE 2 In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. The rock fell $y = 16t^2$ feet during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec.}$$ ■

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas are based on the same limit.

> The following are all interpretations for the limit of the difference quotient
>
> $$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
>
> 1. The slope of the graph of $y = f(x)$ at $x = x_0$
> 2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
> 3. The rate of change of $f(x)$ with respect to x at the $x = x_0$
> 4. The derivative $f'(x_0)$ at $x = x_0$

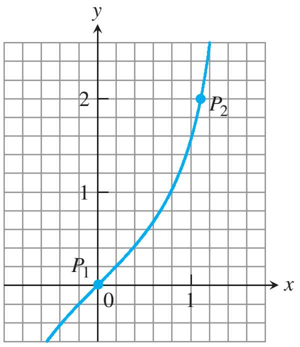
In the next sections, we allow the point x_0 to vary across the domain of the function f.

EXERCISES 3.1

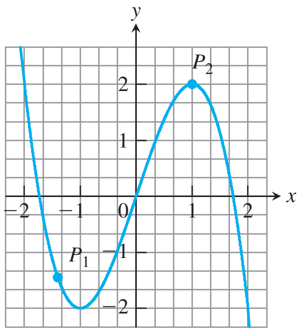
Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y-units per x-unit) at the points P_1 and P_2.

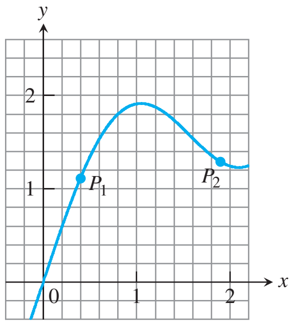
1.



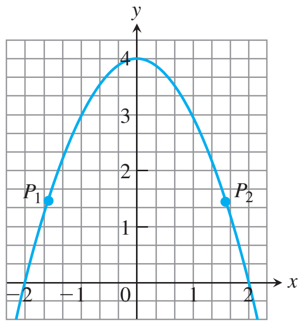
2.



3.



4.



In Exercises 5–10, find an equation for the tangent line to the curve at the given point. Then sketch the curve and tangent line together.

5. $y = 4 - x^2, \quad (-1, 3)$ **6.** $y = (x - 1)^2 + 1, \quad (1, 1)$

7. $y = 2\sqrt{x}, \quad (1, 2)$ **8.** $y = \frac{1}{x^2}, \quad (-1, 1)$

9. $y = x^3, \quad (-2, -8)$ **10.** $y = \frac{1}{x^3}, \quad \left(-2, -\frac{1}{8}\right)$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1, \quad (2, 5)$ **12.** $f(x) = x - 2x^2, \quad (1, -1)$

13. $g(x) = \frac{x}{x - 2}, \quad (3, 3)$ **14.** $g(x) = \frac{8}{x^2}, \quad (2, 2)$

15. $h(t) = t^3, \quad (2, 8)$ **16.** $h(t) = t^3 + 3t, \quad (1, 4)$

17. $f(x) = \sqrt{x}, \quad (4, 2)$ **18.** $f(x) = \sqrt{x + 1}, \quad (8, 3)$

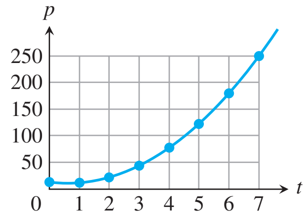
In Exercises 19–22, find the slope of the curve at the point indicated.

19. $y = 5x - 3x^2, \quad x = 1$ **20.** $y = x^3 - 2x + 7, \quad x = -2$

21. $y = \frac{1}{x - 1}, \quad x = 3$ **22.** $y = \frac{x - 1}{x + 1}, \quad x = 0$

Interpreting Derivative Values

23. Growth of yeast cells In a controlled laboratory experiment, yeast cells are grown in an automated cell culture system that counts the number P of cells present at hourly intervals. The number after t hours is shown in the accompanying figure.

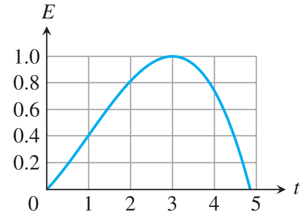


a. Explain what is meant by the derivative $P'(5)$. What are its units?

b. Which is larger, $P'(2)$ or $P'(3)$? Give a reason for your answer.

c. The quadratic curve capturing the trend of the data points (see Section 1.4) is given by $P(t) = 6.10t^2 - 9.28t + 16.43$. Find the instantaneous rate of growth when $t = 5$ hours.

24. Effectiveness of a drug On a scale from 0 to 1, the effectiveness E of a pain-killing drug t hours after entering the bloodstream is displayed in the accompanying figure.



a. At what times does the effectiveness appear to be increasing? What is true about the derivative at those times?

b. At what time would you estimate that the drug reaches its maximum effectiveness? What is true about the derivative at that time? What is true about the derivative as time increases in the 1 hour *before* your estimated time?

At what points do the graphs of the functions in Exercises 25 and 26 have horizontal tangent lines?

25. $f(x) = x^2 + 4x - 1$ **26.** $g(x) = x^3 - 3x$

27. Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x - 1)$.

28. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

29. Object dropped from a tower An object is dropped from the top of a 100-m-high tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

30. Speed of a rocket At t sec after liftoff, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing 10 sec after liftoff?

31. Circle's changing area What is the rate of change of the area of a circle $(A = \pi r^2)$ with respect to the radius when the radius is $r = 3$?

32. Ball's changing volume What is the rate of change of the volume of a ball $(V = (4/3)\pi r^3)$ with respect to the radius when the radius is $r = 2$?

33. Show that the line $y = mx + b$ is its own tangent line at any point $(x_0, mx_0 + b)$.

34. Find the slope of the tangent line to the curve $y = 1/\sqrt{x}$ at the point where $x = 4$.

Testing for Tangent Lines

35. Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent line at the origin? Give reasons for your answer.

36. Does the graph of

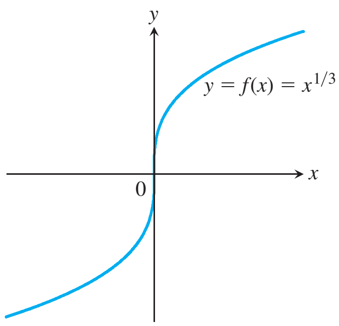
$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent line at the origin? Give reasons for your answer.

Vertical Tangent Lines

We say that a continuous curve $y = f(x)$ has a **vertical tangent line** at the point where $x = x_0$ if the limit of the difference quotient is ∞ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent line at $x = 0$ (see accompanying figure):

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty.$$

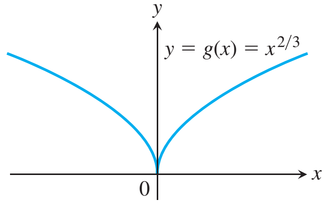


VERTICAL TANGENT LINE AT ORIGIN

However, $y = x^{2/3}$ has *no* vertical tangent line at $x = 0$ (see next figure):

$$\lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{h^{2/3} - 0}{h}$$
$$= \lim_{h\to 0} \frac{1}{h^{1/3}}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



NO VERTICAL TANGENT LINE AT ORIGIN

37. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent line at the origin? Give reasons for your answer.

38. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

have a vertical tangent line at the point (0, 1)? Give reasons for your answer.

T Graph the curves in Exercises 39–48.

a. Where do the graphs appear to have vertical tangent lines?

b. Confirm your findings in part (a) with limit calculations. But before you do, read the introduction to Exercises 37 and 38.

39. $y = x^{2/5}$

40. $y = x^{4/5}$

41. $y = x^{1/5}$

42. $y = x^{3/5}$

43. $y = 4x^{2/5} - 2x$

44. $y = x^{5/3} - 5x^{2/3}$

45. $y = x^{2/3} - (x - 1)^{1/3}$

46. $y = x^{1/3} + (x - 1)^{1/3}$

47. $y = \begin{cases} -\sqrt{|x|}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$

48. $y = \sqrt{|4 - x|}$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 49–52:

a. Plot $y = f(x)$ over the interval $(x_0 - 1/2) \leq x \leq (x_0 + 3)$.

b. Holding x_0 fixed, the difference quotient

$$q(h) = \frac{f(x_0 + h) - f(x_0)}{h}$$

at x_0 becomes a function of the step size h. Enter this function into your CAS workspace.

c. Find the limit of q as $h \to 0$.

d. Define the secant lines $y = f(x_0) + q \cdot (x - x_0)$ for $h = 3, 2,$ and 1. Graph them together with f and the tangent line over the interval in part (a).

49. $f(x) = x^3 + 2x, \quad x_0 = 0$

50. $f(x) = x + \frac{5}{x}, \quad x_0 = 1$

51. $f(x) = x + \sin(2x), \quad x_0 = \pi/2$

52. $f(x) = \cos x + 4\sin(2x), \quad x_0 = \pi$

3.2 The Derivative as a Function

HISTORICAL ESSAY
The Derivative
www.goo.gl/1lNWop

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

We use the notation $f(x)$ in the definition, rather than $f(x_0)$ as before, to emphasize that f' is a function of the independent variable x with respect to which the derivative function $f'(x)$ is being defined. The domain of f' is the set of points in the domain of f for