## Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

> **THEOREM 1—Differentiability Implies Continuity** If $f$ has a derivative at $x = c$, then $f$ is continuous at $x = c$.

**Proof** Given that $f'(c)$ exists, we must show that $\lim_{x \to c} f(x) = f(c)$, or equivalently, that $\lim_{h \to 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$f(c + h) = f(c) + (f(c + h) - f(c)) \qquad \text{Add and subtract } f(c).$$

$$= f(c) + \frac{f(c + h) - f(c)}{h} \cdot h. \qquad \text{Divide and multiply by } h.$$

Now take limits as $h \to 0$. By Theorem 1 of Section 2.2,

$$\lim_{h \to 0} f(c + h) = \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{h \to 0} h$$

$$= f(c) + f'(c) \cdot 0$$

$$= f(c) + 0$$

$$= f(c).$$

■

Similar arguments with one-sided limits show that if $f$ has a derivative from one side (right or left) at $x = c$, then $f$ is continuous from that side at $x = c$.

Theorem 1 says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there. The greatest integer function $y = \lfloor x \rfloor$ fails to be differentiable at every integer $x = n$ (Example 4, Section 2.5).

**Caution** The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw with the absolute value function in Example 4. ●

## EXERCISES 3.2

### Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2;\quad f'(-3), f'(0), f'(1)$
2. $F(x) = (x - 1)^2 + 1;\quad F'(-1), F'(0), F'(2)$
3. $g(t) = \dfrac{1}{t^2};\quad g'(-1), g'(2), g'\left(\sqrt{3}\right)$
4. $k(z) = \dfrac{1 - z}{2z};\quad k'(-1), k'(1), k'\left(\sqrt{2}\right)$
5. $p(\theta) = \sqrt{3\theta};\quad p'(1), p'(3), p'(2/3)$
6. $r(s) = \sqrt{2s + 1};\quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\dfrac{dy}{dx}$ if $y = 2x^3$
8. $\dfrac{dr}{ds}$ if $r = s^3 - 2s^2 + 3$
9. $\dfrac{ds}{dt}$ if $s = \dfrac{t}{2t + 1}$
10. $\dfrac{dv}{dt}$ if $v = t - \dfrac{1}{t}$
11. $\dfrac{dp}{dq}$ if $p = q^{3/2}$
12. $\dfrac{dz}{dw}$ if $z = \dfrac{1}{\sqrt{w^2 - 1}}$

### Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

13. $f(x) = x + \dfrac{9}{x},\quad x = -3$
14. $k(x) = \dfrac{1}{2 + x},\quad x = 2$
15. $s = t^3 - t^2,\quad t = -1$
16. $y = \dfrac{x + 3}{1 - x},\quad x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \dfrac{8}{\sqrt{x - 2}},\quad (x, y) = (6, 4)$
18. $w = g(z) = 1 + \sqrt{4 - z},\quad (z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

19. $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$
20. $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \dfrac{1}{x}$
21. $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4 - \theta}}$
22. $\left.\dfrac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

## Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

**23.** $f(x) = \dfrac{1}{x + 2}$

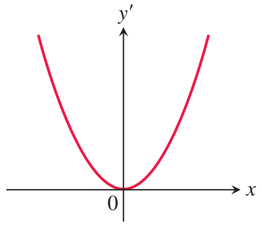
**24.** $f(x) = x^2 - 3x + 4$

**25.** $g(x) = \dfrac{x}{x - 1}$

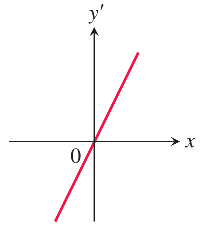
**26.** $g(x) = 1 + \sqrt{x}$

## Graphs

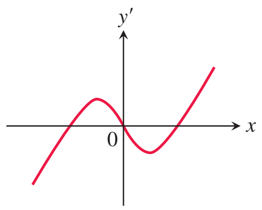
Match the functions graphed in Exercises 27–30 with the derivatives graphed in the accompanying figures (a)–(d).



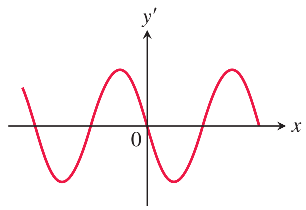
(a)



(b)

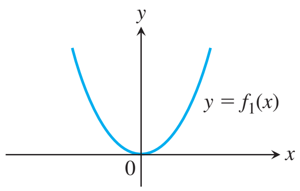


(c)

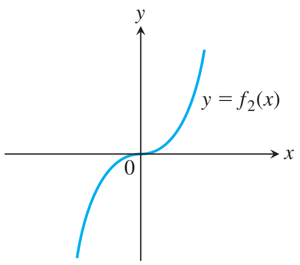


(d)

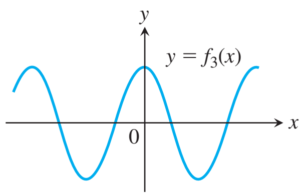
**27.**



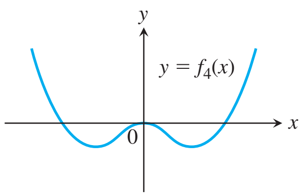

**28.**



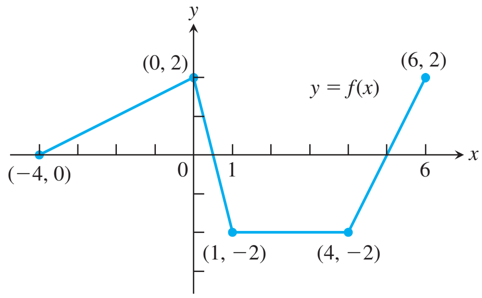

**29.**




**30.**




**31. a.** The graph in the accompanying figure is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is $f'$ not defined? Give reasons for your answer.




**b.** Graph the derivative of $f$.
The graph should show a step function.

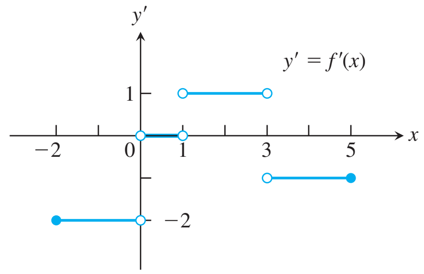
**32. Recovering a function from its derivative**

**a.** Use the following information to graph the function $f$ over the closed interval $[-2, 5]$.

**i)** The graph of $f$ is made of closed line segments joined end to end.

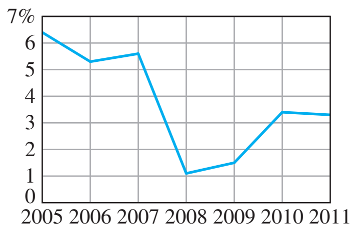
**ii)** The graph starts at the point $(-2, 3)$.

**iii)** The derivative of $f$ is the step function in the figure shown here.



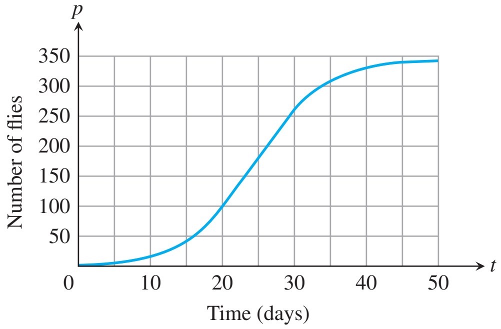

**b.** Repeat part (a), assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

**33. Growth in the economy** The graph in the accompanying figure shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 2005–2011. Graph $dy/dt$ (where defined).



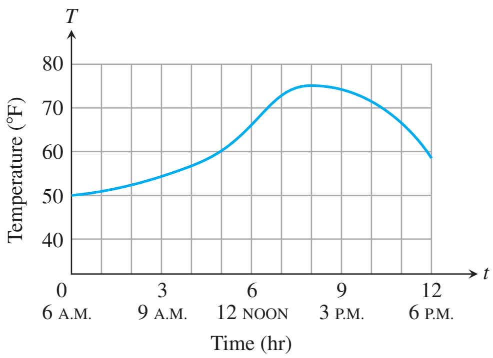

**34. Fruit flies** (*Continuation of Example 4, Section 2.1.*) Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

**a.** Use the graphical technique of Example 3 to graph the derivative of the fruit fly population. The graph of the population is reproduced here.




**b.** During what days does the population seem to be increasing fastest? Slowest?

**35. Temperature** The given graph shows the outside temperature $T$ in °F, between 6 A.M. and 6 P.M.



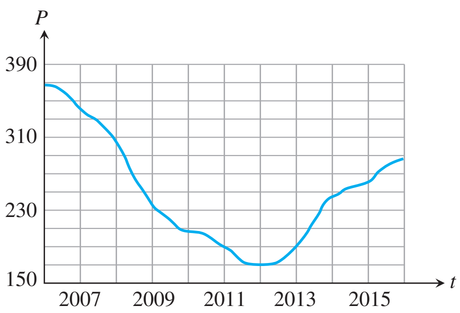

**a.** Estimate the rate of temperature change at the times

**i)** 7 A.M. **ii)** 9 A.M. **iii)** 2 P.M. **iv)** 4 P.M.

**b.** At what time does the temperature increase most rapidly? Decrease most rapidly? What is the rate for each of those times?

**c.** Use the graphical technique of Example 3 to graph the derivative of temperature $T$ versus time $t$.

**36.** Average single-family home prices $P$ (in thousands of dollars) in Sacramento, California, are shown in the accompanying figure from 2006 through 2015.




**a.** During what years did home prices decrease? increase?

**b.** Estimate home prices at the end of

**i.** 2007 **ii.** 2012 **iii.** 2015

**c.** Estimate the rate of change of home prices at the beginning of

**i.** 2007 **ii.** 2010 **iii.** 2014

**d.** During what year did home prices drop most rapidly and what is an estimate of this rate?

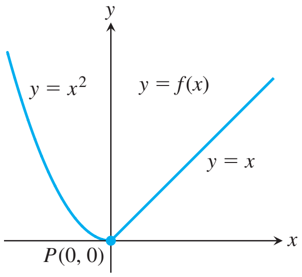
**e.** During what year did home prices rise most rapidly and what is an estimate of this rate?

**f.** Use the graphical technique of Example 3 to graph the derivative of home price $P$ versus time $t$.

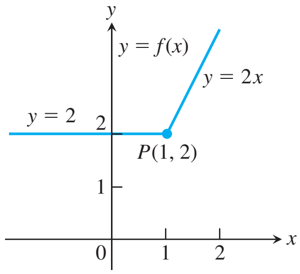
## One-Sided Derivatives

Compute the right-hand and left-hand derivatives as limits to show that the functions in Exercises 37–40 are not differentiable at the point $P$.

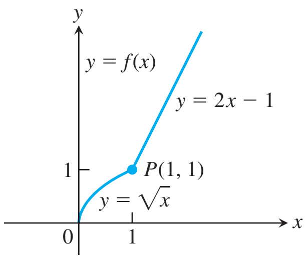
**37.**



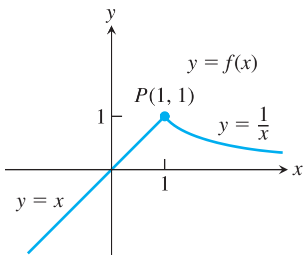

**38.**




**39.**




**40.**




In Exercises 41–44, determine if the piecewise-defined function is differentiable at the origin.

**41.** $f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$

**42.** $g(x) = \begin{cases} x^{2/3}, & x \geq 0 \\ x^{1/3}, & x < 0 \end{cases}$

**43.** $f(x) = \begin{cases} 2x + \tan x, & x \geq 0 \\ x^2, & x < 0 \end{cases}$

**44.** $g(x) = \begin{cases} 2x - x^3 - 1, & x \geq 0 \\ x - \dfrac{1}{x + 1}, & x < 0 \end{cases}$

## Differentiability and Continuity on an Interval

Each figure in Exercises 45–50 shows the graph of a function over a closed interval D. At what domain points does the function appear to be

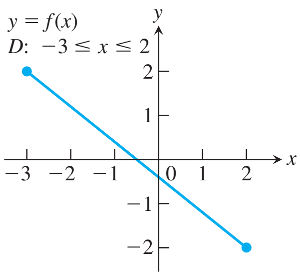
**a.** differentiable?

**b.** continuous but not differentiable?

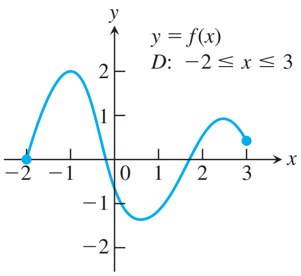
**c.** neither continuous nor differentiable?

Give reasons for your answers.

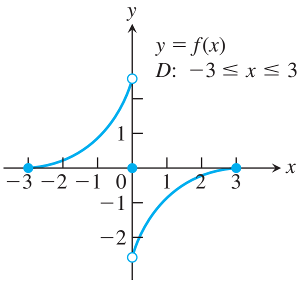
45.



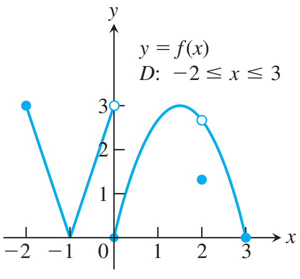

46.



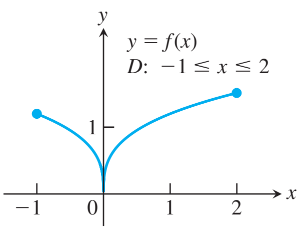

47.



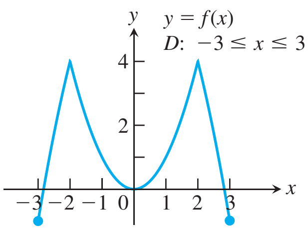

48.




49.




50.




## Theory and Examples

In Exercises 51–54,

**a.** Find the derivative $f'(x)$ of the given function $y = f(x)$.

**b.** Graph $y = f(x)$ and $y = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.

**c.** For what values of $x$, if any, is $f'$ positive? Zero? Negative?

**d.** Over what intervals of $x$-values, if any, does the function $y = f(x)$ increase as $x$ increases? Decrease as $x$ increases? How is this related to what you found in part (c)? (We will say more about this relationship in Section 4.3.)

**51.** $y = -x^2$ **52.** $y = -1/x$

**53.** $y = x^3/3$ **54.** $y = x^4/4$

**55. Tangent to a parabola** Does the parabola $y = 2x^2 - 13x + 5$ have a tangent line whose slope is $-1$? If so, find an equation for the line and the point of tangency. If not, why not?

**56. Tangent to $y = \sqrt{x}$** Does any tangent line to the curve $y = \sqrt{x}$ cross the $x$-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

**57. Derivative of $-f$** Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

**58. Derivative of multiples** Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

**59. Limit of a quotient** Suppose that functions $g(t)$ and $h(t)$ are defined for all values of $t$ and $g(0) = h(0) = 0$. Can $\lim_{t\to 0}(g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

**60. a.** Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that $f$ is differentiable at $x = 0$ and find $f'(0)$.

**b.** Show that

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

T **61.** Graph $y = 1/\left(2\sqrt{x}\right)$ in a window that has $0 \le x \le 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

T **62.** Graph $y = 3x^2$ in a window that has $-2 \le x \le 2$, $0 \le y \le 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

**63. Derivative of $y = |x|$** Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

T **64. Weierstrass's nowhere differentiable continuous function** The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty}(2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + (2/3)^1 \cos(9\pi x) + (2/3)^2 \cos(9^2 \pi x) + (2/3)^3 \cos(9^3 \pi x) + \cdots + (2/3)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

## COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 65–70.

**a.** Plot $y = f(x)$ to see that function's global behavior.

**b.** Define the difference quotient $q$ at a general point $x$, with general step size $h$.

**c.** Take the limit as $h \to 0$. What formula does this give?

**d.** Substitute the value $x = x_0$ and plot the function $y = f(x)$ together with its tangent line at that point.

**e.** Substitute various values for $x$ larger and smaller than $x_0$ into the formula obtained in part (c). Do the numbers make sense with your picture?

**f.** Graph the formula obtained in part (c). What does it mean when its values are negative? Zero? Positive? Does this make sense with your plot from part (a)? Give reasons for your answer.