EXERCISES 3.3

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$

2. $y = x^2 + x + 8$

3. $s = 5t^3 - 3t^5$

4. $w = 3z^7 - 7z^3 + 21z^2$

5. $y = \dfrac{4x^3}{3} - x + 2e^x$

6. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + e^{-x}$

7. $w = 3z^{-2} - \dfrac{1}{z}$

8. $s = -2t^{-1} + \dfrac{4}{t^2}$

9. $y = 6x^2 - 10x - 5x^{-2}$

10. $y = 4 - 2x - x^{-3}$

11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$

12. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$

In Exercises 13–16, find y' **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$

14. $y = (2x + 3)(5x^2 - 4x)$

15. $y = (x^2 + 1)\left(x + 5 + \dfrac{1}{x}\right)$

16. $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 17–40.

17. $y = \dfrac{2x + 5}{3x - 2}$

18. $z = \dfrac{4 - 3x}{3x^2 + x}$

19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$

20. $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$

21. $v = (1 - t)(1 + t^2)^{-1}$

22. $w = (2x - 7)^{-1}(x + 5)$

23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$

24. $u = \dfrac{5x + 1}{2\sqrt{x}}$

25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$

26. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$

27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$

28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

29. $y = 2e^{-x} + e^{3x}$

30. $y = \dfrac{x^2 + 3e^x}{2e^x - x}$

31. $y = x^3e^x$

32. $w = re^{-r}$

33. $y = x^{9/4} + e^{-2x}$

34. $y = x^{-3/5} + \pi^{3/2}$

35. $s = 2t^{3/2} + 3e^2$

36. $w = \dfrac{1}{z^{1.4}} + \dfrac{\pi}{\sqrt{z}}$

37. $y = \sqrt[7]{x^2} - x^e$

38. $y = \sqrt[3]{x^{9.6}} + 2e^{1.3}$

39. $r = \dfrac{e^s}{s}$

40. $r = e^{\theta}\left(\dfrac{1}{\theta^2} + \theta^{-\pi/2}\right)$

Find the derivatives of all orders of the functions in Exercises 41–44.

41. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$

42. $y = \dfrac{x^5}{120}$

43. $y = (x - 1)(x + 2)(x + 3)$

44. $y = (4x^2 + 3)(2 - x)x$

Find the first and second derivatives of the functions in Exercises 45–52.

45. $y = \dfrac{x^3 + 7}{x}$

46. $s = \dfrac{t^2 + 5t - 1}{t^2}$

47. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$

48. $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$

49. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$

50. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

51. $w = 3z^2e^{2z}$

52. $w = e^z(z - 1)(z^2 + 1)$

53. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a. $\dfrac{d}{dx}(uv)$ **b.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c.** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d.** $\dfrac{d}{dx}(7v - 2u)$

54. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a. $\dfrac{d}{dx}(uv)$ **b.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c.** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d.** $\dfrac{d}{dx}(7v - 2u)$

Slopes and Tangent Lines

55. a. Normal line to a curve Find an equation for the line perpendicular to the tangent line to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$.

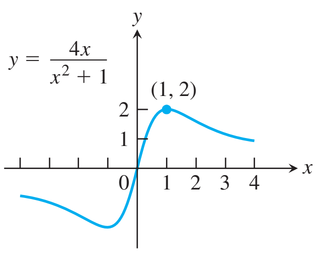
b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

c. Tangent lines having specified slope Find equations for the tangent lines to the curve at the points where the slope of the curve is 8.

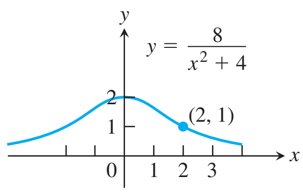
56. a. Horizontal tangent lines Find equations for the horizontal tangent lines to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangent lines at the points of tangency.

b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent line at this point.

57. Find the tangent lines to *Newton's serpentine* (graphed here) at the origin and the point $(1, 2)$.



58. Find the tangent line to the *Witch of Agnesi* (graphed here) at the point $(2, 1)$.



59. Quadratic tangent to identity function The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and is tangent to the line $y = x$ at the origin. Find a, b, and c.

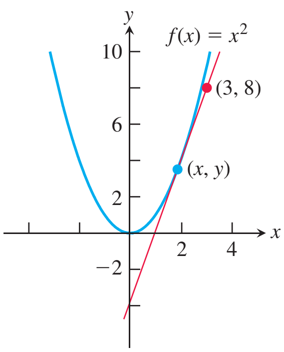
60. Quadratics having a common tangent The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point (1, 0). Find a, b, and c.

61. Find all points (x, y) on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.

62. Find all points (x, y) on the graph of $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1$ with tangent lines parallel to the line $8x - 2y = 1$.

63. Find all points (x, y) on the graph of $y = x/(x - 2)$ with tangent lines perpendicular to the line $y = 2x + 3$.

64. Find all points (x, y) on the graph of $f(x) = x^2$ with tangent lines passing through the point (3, 8).



65. Assume that functions f and g are differentiable with $f(1) = 2$, $f'(1) = -3$, $g(1) = 4$, and $g'(1) = -2$. Find the equation of the line tangent to the graph of $F(x) = f(x)g(x)$ at $x = 1$.

66. Assume that functions f and g are differentiable with $f(2) = 3$, $f'(2) = -1$, $g(2) = -4$, and $g'(2) = 1$. Find an equation of the line perpendicular to the graph of $F(x) = \dfrac{f(x) + 3}{x - g(x)}$ at $x = 2$.

67. a. Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

T **b.** Graph the curve and tangent line together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.

T **c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent line simultaneously.

68. a. Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.

T **b.** Graph the curve and tangent line together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.

T **c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent line simultaneously.

Theory and Examples

For Exercises 69 and 70 evaluate each limit by first converting each to a derivative at a particular x-value.

69. $\displaystyle\lim_{x \to 1} \frac{x^{50} - 1}{x - 1}$

70. $\displaystyle\lim_{x \to -1} \frac{x^{2/9} - 1}{x + 1}$

71. Find the value of a that makes the following function differentiable for all x-values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \geq 0 \end{cases}$$

72. Find the values of a and b that make the following function differentiable for all x-values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \leq -1 \end{cases}$$

73. The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$. Find $P'(x)$.

74. The body's reaction to medicine The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to the medicine. In Section 4.5, we will see how to find the amount of medicine to which the body is most sensitive.

75. Suppose that the function v in the Derivative Product Rule has a constant value c. What does the Derivative Product Rule then say? What does this say about the Derivative Constant Multiple Rule?

76. The Reciprocal Rule

a. The *Reciprocal Rule* says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Derivative Quotient Rule.

b. Show that the Reciprocal Rule and the Derivative Product Rule together imply the Derivative Quotient Rule.

77. Generalizing the Product Rule The Derivative Product Rule gives the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$$

for the derivative of the product uv of two differentiable functions of x.

a. What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x?

b. What is the formula for the derivative of the product $u_1u_2u_3u_4$ of *four* differentiable functions of x?

c. What is the formula for the derivative of a product $u_1u_2u_3 \cdots u_n$ of a finite number n of differentiable functions of x?

78. Power Rule for negative integers Use the Derivative Quotient Rule to prove the Power Rule for negative integers, that is,

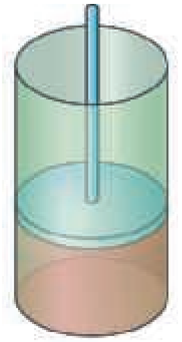
$$\frac{d}{dx}(x^{-m}) = -mx^{-m-1}$$

where m is a positive integer.

79. Cylinder pressure If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a, b, n, and R are constants. Find dP/dV. (See accompanying figure.)



80. The best quantity to order One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, TVs, brooms, or whatever the item might be); k is the cost of placing an order (the same, no matter how often you order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2.

3.4 The Derivative as a Rate of Change

In this section we study applications in which derivatives model the rates at which things change. It is natural to think of a quantity changing with respect to time, but other variables can be treated in the same way. For example, an economist may want to study how the cost of producing steel varies with the number of tons produced, or an engineer may want to know how the power output of a generator varies with its temperature.

Instantaneous Rates of Change

If we interpret the difference quotient $(f(x + h) - f(x))/h$ as the average rate of change in f over the interval from x to $x + h$, we can interpret its limit as $h \to 0$ as the instantaneous rate at which f is changing at the point x. This gives an important interpretation of the derivative.

> **DEFINITION** The **instantaneous rate of change** of f with respect to x at x_0 is the derivative
>
> $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$
>
> provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

EXAMPLE 1 The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4}D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?