EXERCISES 3.6

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = e^{-x}$
5. $y = \sqrt{u}, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = \pi x^2$
8. $y = -\sec u, \quad u = \frac{1}{x} + 7x$

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \sqrt{3x^2 - 4x + 6}$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \tan^3 x$
18. $y = 5\cos^{-4} x$
19. $y = e^{-5x}$
20. $y = e^{2x/3}$
21. $y = e^{5-7x}$
22. $y = e^{(4\sqrt{x}+x^2)}$

Find the derivatives of the functions in Exercises 23–50.

23. $p = \sqrt{3 - t}$
24. $q = \sqrt[3]{2r - r^2}$
25. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$
26. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$
27. $r = (\csc\theta + \cot\theta)^{-1}$
28. $r = 6(\sec\theta - \tan\theta)^{3/2}$
29. $y = x^2\sin^4 x + x\cos^{-2} x$
30. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$
31. $y = \frac{1}{18}(3x - 2)^6 + \left(4 - \frac{1}{2x^2}\right)^{-1}$
32. $y = (5 - 2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$
33. $y = (4x + 3)^4(x + 1)^{-3}$
34. $y = (2x - 5)^{-1}(x^2 - 5x)^6$
35. $y = xe^{-x} + e^{x^3}$
36. $y = (1 + 2x)e^{-2x}$
37. $y = (x^2 - 2x + 2)e^{5x/2}$
38. $y = (9x^2 - 6x + 2)e^{x^3}$
39. $h(x) = x\tan\left(2\sqrt{x}\right) + 7$
40. $k(x) = x^2\sec\left(\frac{1}{x}\right)$
41. $f(x) = \sqrt{7 + x\sec x}$
42. $g(x) = \frac{\tan 3x}{(x + 7)^4}$
43. $f(\theta) = \left(\frac{\sin\theta}{1 + \cos\theta}\right)^2$
44. $g(t) = \left(\frac{1 + \sin 3t}{3 - 2t}\right)^{-1}$
45. $r = \sin(\theta^2)\cos(2\theta)$
46. $r = \sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$
47. $q = \sin\left(\frac{t}{\sqrt{t + 1}}\right)$
48. $q = \cot\left(\frac{\sin t}{t}\right)$
49. $y = \cos\left(e^{-\theta^2}\right)$
50. $y = \theta^3 e^{-2\theta}\cos 5\theta$

In Exercises 51–70, find dy/dt.

51. $y = \sin^2(\pi t - 2)$
52. $y = \sec^2 \pi t$
53. $y = (1 + \cos 2t)^{-4}$
54. $y = (1 + \cot(t/2))^{-2}$
55. $y = (t\tan t)^{10}$
56. $y = (t^{-3/4}\sin t)^{4/3}$
57. $y = e^{\cos^2(\pi t - 1)}$
58. $y = \left(e^{\sin(t/2)}\right)^3$
59. $y = \left(\frac{t^2}{t^3 - 4t}\right)^3$
60. $y = \left(\frac{3t - 4}{5t + 2}\right)^{-5}$
61. $y = \sin(\cos(2t - 5))$
62. $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$
63. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$
64. $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$
65. $y = \sqrt{1 + \cos(t^2)}$
66. $y = 4\sin\left(\sqrt{1 + \sqrt{t}}\right)$
67. $y = \tan^2(\sin^3 t)$
68. $y = \cos^4(\sec^2 3t)$
69. $y = 3t(2t^2 - 5)^4$
70. $y = \sqrt{3t + \sqrt{2 + \sqrt{1 - t}}}$

Second Derivatives

Find y'' in Exercises 71–78.

71. $y = \left(1 + \frac{1}{x}\right)^3$
72. $y = \left(1 - \sqrt{x}\right)^{-1}$
73. $y = \frac{1}{9}\cot(3x - 1)$
74. $y = 9\tan\left(\frac{x}{3}\right)$
75. $y = x(2x + 1)^4$
76. $y = x^2(x^3 - 1)^5$
77. $y = e^{x^2} + 5x$
78. $y = \sin(x^2 e^x)$

For each of the following functions, solve both $f'(x) = 0$ and $f''(x) = 0$ for x.

79. $f(x) = x(x - 4)^3$
80. $f(x) = \sec^2 x - 2\tan x \quad \text{for} \quad 0 \le x \le 2\pi$

Finding Derivative Values

In Exercises 81–86, find the value of $(f \circ g)'$ at the given value of x.

81. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$
82. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$
83. $f(u) = \cot\frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$
84. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$
85. $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$
86. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

87. Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$?

88. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?

89. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $2f(x),\quad x = 2$ **b.** $f(x) + g(x),\quad x = 3$

c. $f(x) \cdot g(x),\quad x = 3$ **d.** $f(x)/g(x),\quad x = 2$

e. $f(g(x)),\quad x = 2$ **f.** $\sqrt{f(x)},\quad x = 2$

g. $1/g^2(x),\quad x = 3$ **h.** $\sqrt{f^2(x) + g^2(x)},\quad x = 2$

90. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $5f(x) - g(x),\quad x = 1$ **b.** $f(x)g^3(x),\quad x = 0$

c. $\dfrac{f(x)}{g(x) + 1},\quad x = 1$ **d.** $f(g(x)),\quad x = 0$

e. $g(f(x)),\quad x = 0$ **f.** $(x^{11} + f(x))^{-2},\quad x = 1$

g. $f(x + g(x)),\quad x = 0$

91. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

92. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Theory and Examples

What happens if you can write a function as a composition in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 93 and 94.

93. Find dy/dx if $y = x$ by using the Chain Rule with y as a composition of

a. $y = (u/5) + 7\quad$ and $\quad u = 5x - 35$

b. $y = 1 + (1/u)\quad$ and $\quad u = 1/(x - 1)$.

94. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composition of

a. $y = u^3\quad$ and $\quad u = \sqrt{x}$

b. $y = \sqrt{u}\quad$ and $\quad u = x^3$.

95. Find the tangent to $y = ((x - 1)/(x + 1))^2$ at $x = 0$.

96. Find the tangent to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.

97. a. Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.

b. Slopes on a tangent curve What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

98. Slopes on sine curves

a. Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.

b. Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.

c. For a given m, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.

d. The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

99. Running machinery too fast Suppose that a piston is moving straight up and down and that its position at time t sec is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why some machinery breaks when you run it too fast.)

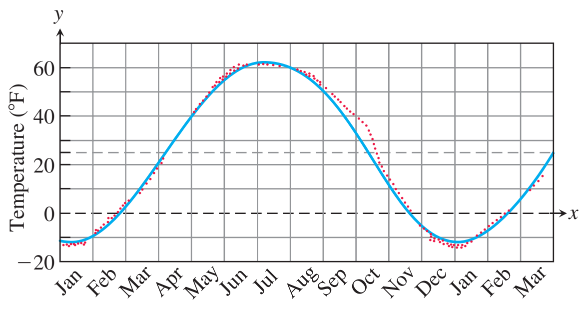
100. Temperatures in Fairbanks, Alaska The graph in the accompanying figure shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25$$

and is graphed in the accompanying figure.

a. On what day is the temperature increasing the fastest?

b. About how many degrees per day is the temperature increasing when it is increasing at its fastest?



101. Particle motion The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

102. Constant acceleration Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s m from its starting point. Show that the body's acceleration is constant.

103. Falling meteorite The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s km from Earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

104. Particle acceleration A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

105. Temperature and the period of a pendulum For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

106. Chain Rule Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the compositions

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

T **107. The derivative of sin 2*x*** Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

108. The derivative of cos(x^2) Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos((x + h)^2) - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

Using the Chain Rule, show that the Power Rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 109 and 110.

109. $x^{1/4} = \sqrt{\sqrt{x}}$

110. $x^{3/4} = \sqrt{x\sqrt{x}}$

111. Consider the function

$$f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right), & x > 0 \\ 0, & x \le 0 \end{cases}$$

a. Show that f is continuous at $x = 0$.

b. Determine f' for $x \ne 0$.

c. Show that f is not differentiable at $x = 0$.

112. Consider the function

$$f(x) = \begin{cases} x^2\cos\left(\dfrac{2}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

a. Show that f is continuous at $x = 0$.

b. Determine f' for $x \ne 0$.

c. Show that f is differentiable at $x = 0$.

d. Show that f' is not continuous at $x = 0$.

113. Verify each of the following statements.

a. If f is even, then f' is odd.

b. If f is odd, then f' is even.

COMPUTER EXPLORATIONS

Trigonometric Polynomials

114. As the accompanying figure shows, the trigonometric "polynomial"

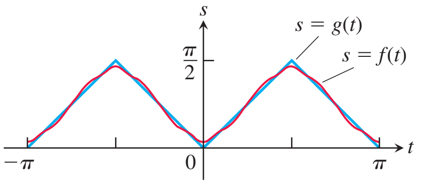
$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t - 0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

a. Graph dg/dt (where defined) over $[-\pi, \pi]$.

b. Find df/dt.

c. Graph df/dt. Where does the approximation of dg/dt by df/dt seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



115. (*Continuation of Exercise 114.*) In Exercise 114, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the trigonometric "polynomial"

$$s = h(t) = 1.2732\sin 2t + 0.4244\sin 6t + 0.25465\sin 10t + 0.18189\sin 14t + 0.14147\sin 18t$$