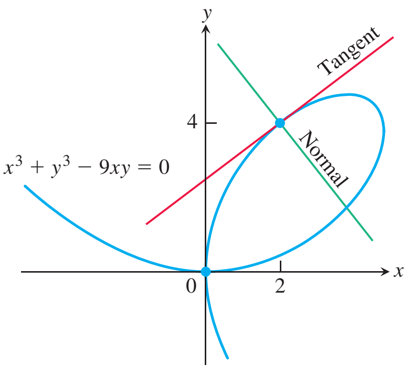


FIGURE 3.34 Example 5 shows how to find equations for the tangent and normal to the folium of Descartes at (2,4).

EXAMPLE 5 Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.34).

Solution The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$ Differentiate both sides with respect to x.

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$ Treat xy as a product and y as a function of x.

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}.$$ Solve for dy/dx.

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope $4/5$:

$$y = 4 + \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x + \frac{12}{5}.$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope $-5/4$:

$$y = 4 - \frac{5}{4}(x - 2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}.$$

EXERCISES 3.7

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–16.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $2xy + y^2 = x + y$
4. $x^3 - xy + y^3 = 1$
5. $x^2(x - y)^2 = x^2 - y^2$
6. $(3xy + 7)^2 = 6y$
7. $y^2 = \dfrac{x - 1}{x + 1}$
8. $x^3 = \dfrac{2x - y}{x + 3y}$
9. $x = \sec y$
10. $xy = \cot(xy)$
11. $x + \tan(xy) = 0$
12. $x^4 + \sin y = x^3y^2$
13. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$
14. $x\cos(2x + 3y) = y \sin x$
15. $e^{2x} = \sin(x + 3y)$
16. $e^{x^2y} = 2x + 2y$

Find $dr/d\theta$ in Exercises 17–20.

17. $\theta^{1/2} + r^{1/2} = 1$
18. $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$
19. $\sin(r\theta) = \dfrac{1}{2}$
20. $\cos r + \cot\theta = e^{r\theta}$

Second Derivatives

In Exercises 21–28, use implicit differentiation to find dy/dx and then d^2y/dx^2. Write the solutions in terms of x and y only.

21. $x^2 + y^2 = 1$

22. $x^{2/3} + y^{2/3} = 1$

23. $y^2 = e^{x^2} + 2x$

24. $y^2 - 2x = 1 - 2y$

25. $2\sqrt{y} = x - y$

26. $xy + y^2 = 1$

27. $3 + \sin y = y - x^3$

28. $\ln y = xe^y - 2$

29. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.

30. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

In Exercises 31 and 32, find the slope of the curve at the given points.

31. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$

32. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

Slopes, Tangents, and Normals

In Exercises 33–42, verify that the given point is on the curve and find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

33. $x^2 + xy - y^2 = 1, \quad (2, 3)$

34. $x^2 + y^2 = 25, \quad (3, -4)$

35. $x^2y^2 = 9, \quad (-1, 3)$

36. $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$

37. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$

38. $x^2 - \sqrt{3}xy + 2y^2 = 5, \quad \left(\sqrt{3}, 2\right)$

39. $2xy + \pi \sin y = 2\pi, \quad (1, \pi/2)$

40. $x \sin 2y = y \cos 2x, \quad (\pi/4, \pi/2)$

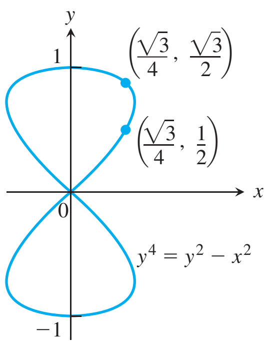
41. $y = 2\sin(\pi x - y), \quad (1, 0)$

42. $x^2 \cos^2 y - \sin y = 0, \quad (0, \pi)$

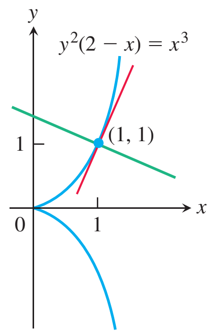
43. Parallel tangents Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

44. Normals parallel to a line Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

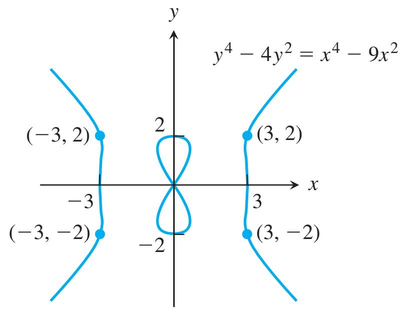
45. The eight curve Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



46. The cissoid of Diocles (from about 200 B.C.) Find equations for the tangent and normal to the cissoid of Diocles $y^2(2 - x) = x^3$ at $(1, 1)$.



47. The devil's curve (Gabriel Cramer, 1750) Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



48. The folium of Descartes (See Figure 3.29)

a. Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points $(4, 2)$ and $(2, 4)$.

b. At what point other than the origin does the folium have a horizontal tangent?

c. Find the coordinates of the point A in Figure 3.29 where the folium has a vertical tangent.

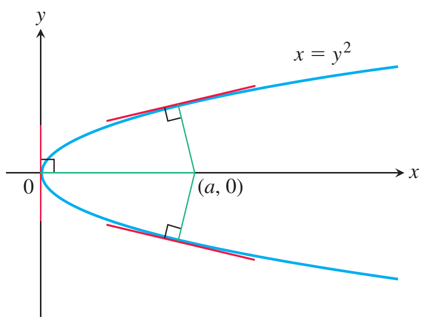
Theory and Examples

49. Intersecting normal The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

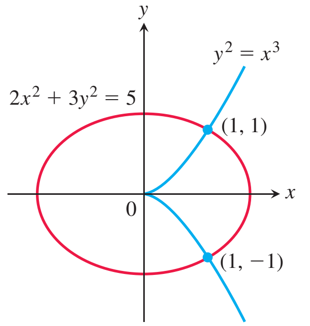
50. Power rule for rational exponents Let p and q be integers with $q > 0$. If $y = x^{p/q}$, differentiate the equivalent equation $y^q = x^p$ implicitly and show that, for $y \neq 0$,

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

51. Normals to a parabola Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown in the accompanying diagram, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



52. Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.

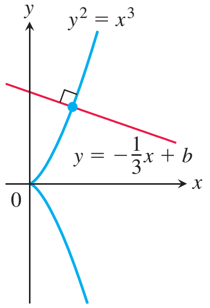


53. Verify that the following pairs of curves meet orthogonally.

a. $x^2 + y^2 = 4, \quad x^2 = 3y^2$

b. $x = 1 - y^2, \quad x = \frac{1}{3}y^2$

54. The graph of $y^2 = x^3$ is called a **semicubical parabola** and is shown in the accompanying figure. Determine the constant b so that the line $y = -\frac{1}{3}x + b$ meets this graph orthogonally.



T In Exercises 55 and 56, find both dy/dx (treating y as a differentiable function of x) and dx/dy (treating x as a differentiable function of y). How do dy/dx and dx/dy seem to be related? Explain the relationship geometrically in terms of the graphs.

55. $xy^3 + x^2y = 6$ **56.** $x^3 + y^2 = \sin^2 y$

57. Derivative of arcsine Assume that $y = \sin^{-1} x$ is a differentiable function of x. By differentiating the equation $x = \sin y$ implicitly, show that $dy/dx = 1/\sqrt{1 - x^2}$.

58. Use the formula in Exercise 57 to find dy/dx if

a. $y = (\sin^{-1} x)^2$ **b.** $y = \sin^{-1}\left(\frac{1}{x}\right)$.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 59–66.

a. Plot the equation with the implicit plotter of a CAS. Check to see that the given point P satisfies the equation.

b. Using implicit differentiation, find a formula for the derivative dy/dx and evaluate it at the given point P.

c. Use the slope found in part (b) to find an equation for the tangent line to the curve at P. Then plot the implicit curve and tangent line together on a single graph.

59. $x^3 - xy + y^3 = 7, \quad P(2, 1)$

60. $x^5 + y^3x + yx^2 + y^4 = 4, \quad P(1, 1)$

61. $y^2 + y = \dfrac{2 + x}{1 - x}, \quad P(0, 1)$ **62.** $y^3 + \cos xy = x^2, \quad P(1, 0)$

63. $x + \tan\left(\dfrac{y}{x}\right) = 2, \quad P\left(1, \dfrac{\pi}{4}\right)$

64. $xy^3 + \tan(x + y) = 1, \quad P\left(\dfrac{\pi}{4}, 0\right)$

65. $2y^2 + (xy)^{1/3} = x^2 + 2, \quad P(1, 1)$

66. $x\sqrt{1 + 2y} + y = x^2, \quad P(1, 0)$

3.8 Derivatives of Inverse Functions and Logarithms

In Section 1.6 we saw how the inverse of a function undoes, or inverts, the effect of that function. We defined there the natural logarithm function $f^{-1}(x) = \ln x$ as the inverse of the natural exponential function $f(x) = e^x$. This is one of the most important function-inverse pairs in mathematics and science. We learned how to differentiate the exponential function in Section 3.3. Here we develop a rule for differentiating the inverse of a differentiable function and we apply the rule to find the derivative of the natural logarithm function.

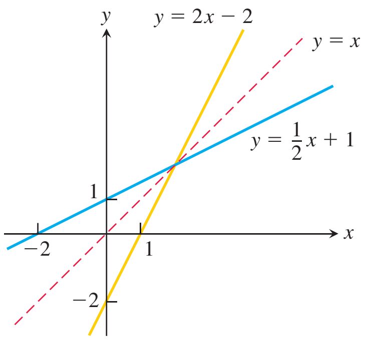


FIGURE 3.35 Graphing a line and its inverse together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other.

Derivatives of Inverses of Differentiable Functions

We calculated the inverse of the function $f(x) = (1/2)x + 1$ to be $f^{-1}(x) = 2x - 2$ in Example 3 of Section 1.6. Figure 3.35 shows the graphs of both functions. If we calculate their derivatives, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$