EXERCISES 3.8

Derivatives of Inverse Functions

In Exercises 1–4:

a. Find $f^{-1}(x)$.

b. Graph f and f^{-1} together.

c. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

1. $f(x) = 2x + 3, \quad a = -1$

2. $f(x) = \dfrac{x + 2}{1 - x}, \quad a = \dfrac{1}{2}$

3. $f(x) = 5 - 4x, \quad a = 1/2$

4. $f(x) = 2x^2, \quad x \ge 0, \quad a = 5$

5. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

b. Graph f and g over an x-interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ (four tangents in all).

d. What lines are tangent to the curves at the origin?

6. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

b. Graph h and k over an x-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs at h and k at $(2, 2)$ and $(-2, -2)$.

d. What lines are tangent to the curves at the origin?

7. Let $f(x) = x^3 - 3x^2 - 1, x \ge 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

8. Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

9. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.

10. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

Derivatives of Logarithms

In Exercises 11–40, find the derivative of y with respect to x, t, or θ, as appropriate.

11. $y = \ln 3x + x$

12. $y = \dfrac{1}{\ln 3x}$

13. $y = \ln(t^2)$

14. $y = \ln(t^{3/2}) + \sqrt{t}$

15. $y = \ln \dfrac{3}{x}$

16. $y = \ln(\sin x)$

17. $y = \ln(\theta + 1) - e^\theta$

18. $y = (\cos\theta)\ln(2\theta + 2)$

19. $y = \ln x^3$

20. $y = (\ln x)^3$

21. $y = t(\ln t)^2$

22. $y = t\ln\sqrt{t}$

23. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$

24. $y = (x^2 \ln x)^4$

25. $y = \dfrac{\ln t}{t}$

26. $y = \dfrac{t}{\sqrt{\ln t}}$

27. $y = \dfrac{\ln x}{1 + \ln x}$

28. $y = \dfrac{x\ln x}{1 + \ln x}$

29. $y = \ln(\ln x)$

30. $y = \ln(\ln(\ln x))$

31. $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$

32. $y = \ln(\sec\theta + \tan\theta)$

33. $y = \ln\dfrac{1}{x\sqrt{x + 1}}$

34. $y = \dfrac{1}{2}\ln\dfrac{1 + x}{1 - x}$

35. $y = \dfrac{1 + \ln t}{1 - \ln t}$

36. $y = \sqrt{\ln\sqrt{t}}$

37. $y = \ln(\sec(\ln\theta))$

38. $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$

39. $y = \ln\left(\dfrac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$

40. $y = \ln\sqrt{\dfrac{(x + 1)^5}{(x + 2)^{20}}}$

Logarithmic Differentiation

In Exercises 41–54, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

41. $y = \sqrt{x(x + 1)}$

42. $y = \sqrt{(x^2 + 1)(x - 1)^2}$

43. $y = \sqrt{\dfrac{t}{t + 1}}$

44. $y = \sqrt{\dfrac{1}{t(t + 1)}}$

45. $y = (\sin\theta)\sqrt{\theta + 3}$

46. $y = (\tan\theta)\sqrt{2\theta + 1}$

47. $y = t(t + 1)(t + 2)$

48. $y = \dfrac{1}{t(t + 1)(t + 2)}$

49. $y = \dfrac{\theta + 5}{\theta\cos\theta}$

50. $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

51. $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$

52. $y = \sqrt{\dfrac{(x + 1)^{10}}{(2x + 1)^5}}$

53. $y = \sqrt[3]{\dfrac{x(x - 2)}{x^2 + 1}}$

54. $y = \sqrt[3]{\dfrac{x(x + 1)(x - 2)}{(x^2 + 1)(2x + 3)}}$

Finding Derivatives

In Exercises 55–62, find the derivative of y with respect to x, t, or θ, as appropriate.

55. $y = \ln(\cos^2\theta)$

56. $y = \ln(3\theta e^{-\theta})$

57. $y = \ln(3te^{-t})$

58. $y = \ln(2e^{-t}\sin t)$

59. $y = \ln\left(\dfrac{e^\theta}{1 + e^\theta}\right)$

60. $y = \ln\left(\dfrac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

61. $y = e^{(\cos t + \ln t)}$

62. $y = e^{\sin t}(\ln t^2 + 1)$

In Exercises 63–66, find dy/dx.

63. $\ln y = e^y \sin x$

64. $\ln xy = e^{x+y}$

65. $x^y = y^x$

66. $\tan y = e^x + \ln x$

In Exercises 67–88, find the derivative of y with respect to the given independent variable.

67. $y = 2^x$

68. $y = 3^{-x}$

69. $y = 5^{\sqrt{s}}$

70. $y = 2^{(s^2)}$

71. $y = x^{\pi}$

72. $y = t^{1-e}$

73. $y = \log_2 5\theta$

74. $y = \log_3(1 + \theta \ln 3)$

75. $y = \log_4 x + \log_4 x^2$

76. $y = \log_{25} e^x - \log_5 \sqrt{x}$

77. $y = \log_2 r \cdot \log_4 r$

78. $y = \log_3 r \cdot \log_9 r$

79. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right)$

80. $y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}$

81. $y = \theta \sin(\log_7 \theta)$

82. $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right)$

83. $y = \log_5 e^x$

84. $y = \log_2\left(\frac{x^2e^2}{2\sqrt{x+1}}\right)$

85. $y = 3^{\log_2 t}$

86. $y = 3\log_8(\log_2 t)$

87. $y = \log_2(8t^{\ln 2})$

88. $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

Logarithmic Differentiation with Exponentials

In Exercises 89–100, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

89. $y = (x+1)^x$

90. $y = x^{(x+1)}$

91. $y = (\sqrt{t})^t$

92. $y = t^{\sqrt{t}}$

93. $y = (\sin x)^x$

94. $y = x^{\sin x}$

95. $y = x^{\ln x}$

96. $y = (\ln x)^{\ln x}$

97. $y^x = x^3 y$

98. $x^{\sin y} = \ln y$

99. $x = y^{xy}$

100. $e^y = y^{\ln x}$

Theory and Applications

101. If we write $g(x)$ for $f^{-1}(x)$, Equation (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \quad \text{or} \quad g'(f(a)) \cdot f'(a) = 1.$$

If we then write x for a, we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that f and g are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to x, using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of g and f. What do you find? (This is not a proof of Theorem 3 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

102. Show that $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

103. If $f(x) = x^n$, $n \geq 1$, show from the definition of the derivative that $f'(0) = 0$.

104. Using mathematical induction, show that for $n > 1$

$$\frac{d^n}{dx^n}\ln x = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

COMPUTER EXPLORATIONS

In Exercises 105–112, you will explore some functions and their inverses together with their derivatives and tangent line approximations at specified points. Perform the following steps using your CAS:

a. Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that f is one-to-one over the interval.

b. Solve the equation $y = f(x)$ for x as a function of y, and name the resulting inverse function g.

c. Find the equation for the tangent line to f at the specified point $(x_0, f(x_0))$.

d. Find the equation for the tangent line to g at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 3 to find the slope of this tangent line.

e. Plot the functions f and g, the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

105. $y = \sqrt{3x-2}, \quad \frac{2}{3} \leq x \leq 4, \quad x_0 = 3$

106. $y = \frac{3x+2}{2x-11}, \quad -2 \leq x \leq 2, \quad x_0 = 1/2$

107. $y = \frac{4x}{x^2+1}, \quad -1 \leq x \leq 1, \quad x_0 = 1/2$

108. $y = \frac{x^3}{x^2+1}, \quad -1 \leq x \leq 1, \quad x_0 = 1/2$

109. $y = x^3 - 3x^2 - 1, \quad 2 \leq x \leq 5, \quad x_0 = \frac{27}{10}$

110. $y = 2 - x - x^3, \quad -2 \leq x \leq 2, \quad x_0 = \frac{3}{2}$

111. $y = e^x, \quad -3 \leq x \leq 5, \quad x_0 = 1$

112. $y = \sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad x_0 = 1$

In Exercises 113 and 114, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

113. $y^{1/3} - 1 = (x+2)^3, \quad -5 \leq x \leq 5, \quad x_0 = -3/2$

114. $\cos y = x^{1/5}, \quad 0 \leq x \leq 1, \quad x_0 = 1/2$

3.9 Inverse Trigonometric Functions

We introduced the six basic inverse trigonometric functions in Section 1.6, but focused there on the arcsine and arccosine functions. Here we complete the study of how all six inverse trigonometric functions are defined, graphed, and evaluated, and how their derivatives are computed.