The derivatives of the inverse trigonometric functions are summarized in Table 3.1.

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1 - u^2}} \dfrac{du}{dx}, \quad |u| < 1$
2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1 - u^2}} \dfrac{du}{dx}, \quad |u| < 1$
3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1 + u^2} \dfrac{du}{dx}$
4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1 + u^2} \dfrac{du}{dx}$
5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2 - 1}} \dfrac{du}{dx}, \quad |u| > 1$
6. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2 - 1}} \dfrac{du}{dx}, \quad |u| > 1$

EXERCISES 3.9

Common Values

Use reference triangles in an appropriate quadrant, as in Example 1, to find the angles in Exercises 1–8.

1. a. $\tan^{-1} 1$ **b.** $\arctan(-\sqrt{3})$ **c.** $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

2. a. $\arctan(-1)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

3. a. $\sin^{-1}\left(\dfrac{-1}{2}\right)$ **b.** $\arcsin\left(\dfrac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\dfrac{-\sqrt{3}}{2}\right)$

4. a. $\sin^{-1}\left(\dfrac{1}{2}\right)$ **b.** $\sin^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$ **c.** $\arcsin\left(\dfrac{\sqrt{3}}{2}\right)$

5. a. $\arccos\left(\dfrac{1}{2}\right)$ **b.** $\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

6. a. $\csc^{-1}\sqrt{2}$ **b.** $\csc^{-1}\left(\dfrac{-2}{\sqrt{3}}\right)$ **c.** $\operatorname{arccsc} 2$

7. a. $\sec^{-1}(-\sqrt{2})$ **b.** $\operatorname{arcsec}\left(\dfrac{2}{\sqrt{3}}\right)$ **c.** $\sec^{-1}(-2)$

8. a. $\operatorname{arccot}(-1)$ **b.** $\cot^{-1}(\sqrt{3})$ **c.** $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

Evaluations

Find the values in Exercises 9–12.

9. $\sin\left(\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)\right)$ **10.** $\sec\left(\cos^{-1}\dfrac{1}{2}\right)$

11. $\tan\left(\sin^{-1}\left(-\dfrac{1}{2}\right)\right)$ **12.** $\cot\left(\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)\right)$

Limits

Find the limits in Exercises 13–20. (If in doubt, look at the function's graph.)

13. $\lim_{x\to 1^-} \sin^{-1} x$ **14.** $\lim_{x\to -1^+} \cos^{-1} x$

15. $\lim_{x\to\infty} \tan^{-1} x$ **16.** $\lim_{x\to -\infty} \tan^{-1} x$

17. $\lim_{x\to\infty} \sec^{-1} x$ **18.** $\lim_{x\to -\infty} \sec^{-1} x$

19. $\lim_{x\to\infty} \csc^{-1} x$ **20.** $\lim_{x\to -\infty} \csc^{-1} x$

Finding Derivatives

In Exercises 21–42, find the derivative of y with respect to the appropriate variable.

21. $y = \cos^{-1}(x^2)$ **22.** $y = \cos^{-1}(1/x)$

23. $y = \sin^{-1}\sqrt{2}\,t$ **24.** $y = \sin^{-1}(1 - t)$

25. $y = \sec^{-1}(2s + 1)$ **26.** $y = \sec^{-1} 5s$

27. $y = \csc^{-1}(x^2 + 1), \quad x > 0$ **28.** $y = \csc^{-1}\dfrac{x}{2}$

29. $y = \sec^{-1}\dfrac{1}{t}, \quad 0 < t < 1$ **30.** $y = \sin^{-1}\dfrac{3}{t^2}$

31. $y = \cot^{-1}\sqrt{t}$ **32.** $y = \cot^{-1}\sqrt{t - 1}$

33. $y = \ln(\tan^{-1} x)$ **34.** $y = \tan^{-1}(\ln x)$

35. $y = \csc^{-1}(e^t)$ **36.** $y = \cos^{-1}(e^{-t})$

37. $y = s\sqrt{1 - s^2} + \cos^{-1} s$ **38.** $y = \sqrt{s^2 - 1} - \sec^{-1} s$

39. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$

40. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$ **41.** $y = x\sin^{-1} x + \sqrt{1 - x^2}$

42. $y = \ln(x^2 + 4) - x\tan^{-1}\left(\dfrac{x}{2}\right)$

For problems 43–46 use implicit differentiation to find $\frac{dy}{dx}$ at the given point P.

43. $3\tan^{-1}x + \sin^{-1}y = \frac{\pi}{4}; \quad P(1, -1)$

44. $\sin^{-1}(x + y) + \cos^{-1}(x - y) = \frac{5\pi}{6}; \quad P\left(0, \frac{1}{2}\right)$

45. $y\cos^{-1}(xy) = \frac{-3\sqrt{2}}{4}\pi; \quad P\left(\frac{1}{2}, -\sqrt{2}\right)$

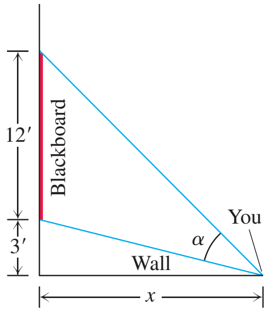
46. $16(\tan^{-1}3y)^2 + 9(\tan^{-1}2x)^2 = 2\pi^2; \quad P\left(\frac{\sqrt{3}}{2}, \frac{1}{3}\right)$

Theory and Examples

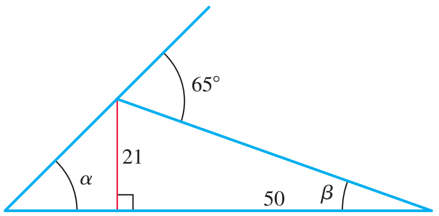
47. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

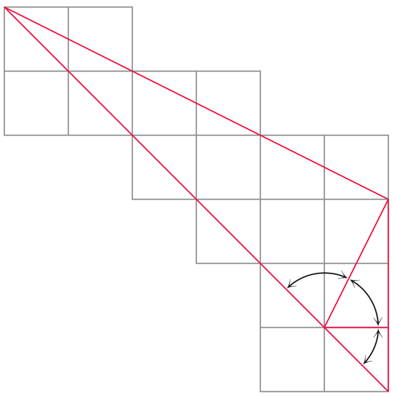
if you are x ft from the front wall.



48. Find the angle α.

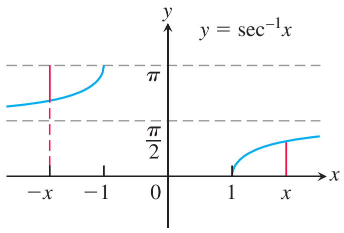


49. Here is an informal proof that $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$. Explain what is going on.



50. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1}x$

a. (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1}x$. See if you can tell what is going on.



b. (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1}x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1}x \qquad \text{Eq. (4), Section 1.6}$$

$$\sec^{-1}x = \cos^{-1}(1/x) \qquad \text{Eq. (1)}$$

Which of the expressions in Exercises 51–54 are defined, and which are not? Give reasons for your answers.

51. **a.** $\tan^{-1}2$ **b.** $\cos^{-1}2$

52. **a.** $\csc^{-1}(1/2)$ **b.** $\csc^{-1}2$

53. **a.** $\sec^{-1}0$ **b.** $\sin^{-1}\sqrt{2}$

54. **a.** $\cot^{-1}(-1/2)$ **b.** $\cos^{-1}(-5)$

55. Use the identity

$$\csc^{-1}u = \frac{\pi}{2} - \sec^{-1}u$$

to derive the formula for the derivative of $\csc^{-1}u$ in Table 3.1 from the formula for the derivative of $\sec^{-1}u$.

56. Derive the formula

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

for the derivative of $y = \tan^{-1}x$ by differentiating both sides of the equivalent equation $\tan y = x$.

57. Use the Derivative Rule in Section 3.8, Theorem 3, to derive

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1.$$

58. Use the identity

$$\cot^{-1}u = \frac{\pi}{2} - \tan^{-1}u$$

to derive the formula for the derivative of $\cot^{-1}u$ in Table 3.1 from the formula for the derivative of $\tan^{-1}u$.

59. What is special about the functions

$$f(x) = \sin^{-1}\frac{x - 1}{x + 1}, \quad x \geq 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$

Explain.

60. What is special about the functions

$$f(x) = \sin^{-1}\frac{1}{\sqrt{x^2 + 1}} \quad \text{and} \quad g(x) = \tan^{-1}\frac{1}{x}?$$

Explain.

T **61.** Find the values of

a. $\sec^{-1} 1.5$ **b.** $\csc^{-1}(-1.5)$ **c.** $\cot^{-1} 2$

T **62.** Find the values of

a. $\sec^{-1}(-3)$ **b.** $\csc^{-1} 1.7$ **c.** $\cot^{-1}(-2)$

T In Exercises 63–65, find the domain and range of each composite function. Then graph the composition of the two functions on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

63. a. $y = \tan^{-1}(\tan x)$ **b.** $y = \tan(\tan^{-1} x)$

64. a. $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1} x)$

65. a. $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1} x)$

T Use your graphing utility for Exercises 66–70.

66. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

67. Newton's serpentine Graph Newton's serpentine, $y = 4x/(x^2 + 1)$. Then graph $y = 2\sin(2\tan^{-1} x)$ in the same graphing window. What do you see? Explain.

68. Graph the rational function $y = (2 - x^2)/x^2$. Then graph $y = \cos(2\sec^{-1} x)$ in the same graphing window. What do you see? Explain.

69. Graph $f(x) = \sin^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

70. Graph $f(x) = \tan^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

3.10 Related Rates

In this section we look at questions that arise when two or more related quantities are changing. The problem of determining how the rate of change of one of them affects the rate of change of the others is called a *related rates problem*.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate both sides with respect to t to find an equation relating the rates of change of V and r,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to directly measure the rate of increase of the volume (the rate at which air is being pumped into the balloon) than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt.

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

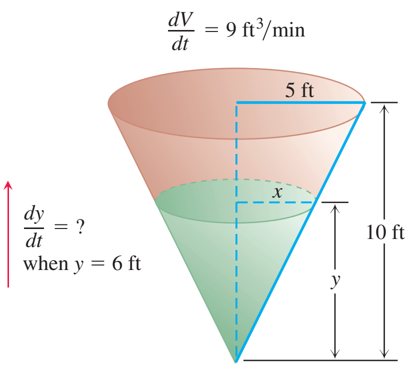


FIGURE 3.45 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

EXAMPLE 1 Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution Figure 3.45 shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.