14. How do derivatives arise in the study of motion? What can you learn about an object's motion along a line by examining the derivatives of the object's position function? Give examples.

15. How can derivatives arise in economics?

16. Give examples of still other applications of derivatives.

17. What do the limits $\lim_{h\to 0}((\sin h)/h)$ and $\lim_{h\to 0}((\cos h - 1)/h)$ have to do with the derivatives of the sine and cosine functions? What *are* the derivatives of these functions?

18. Once you know the derivatives of $\sin x$ and $\cos x$, how can you find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$? What *are* the derivatives of these functions?

19. At what points are the six basic trigonometric functions continuous? How do you know?

20. What is the rule for calculating the derivative of a composition of two differentiable functions? How is such a derivative evaluated? Give examples.

21. If u is a differentiable function of x, how do you find $(d/dx)(u^n)$ if n is an integer? If n is a real number? Give examples.

22. What is implicit differentiation? When do you need it? Give examples.

23. What is the derivative of the natural logarithm function $\ln x$? How does the domain of the derivative compare with the domain of the function?

24. What is the derivative of the exponential function a^x, $a > 0$ and $a \neq 1$? What is the geometric significance of the limit of $(a^h - 1)/h$ as $h \to 0$? What is the limit when a is the number e?

25. What is the derivative of $\log_a x$? Are there any restrictions on a?

26. What is logarithmic differentiation? Give an example.

27. How can you write any real power of x as a power of e? Are there any restrictions on x? How does this lead to the Power Rule for differentiating arbitrary real powers?

28. What is one way of expressing the special number e as a limit? What is an approximate numerical value of e correct to 7 decimal places?

29. What are the derivatives of the inverse trigonometric functions? How do the domains of the derivatives compare with the domains of the functions?

30. How do related rates problems arise? Give examples.

31. Outline a strategy for solving related rates problems. Illustrate with an example.

32. What is the linearization $L(x)$ of a function $f(x)$ at a point $x = a$? What is required of f at a for the linearization to exist? How are linearizations used? Give examples.

33. If x moves from a to a nearby value $a + dx$, how do you estimate the corresponding change in the value of a differentiable function $f(x)$? How do you estimate the relative change? The percentage change? Give an example.

CHAPTER 3 Practice Exercises

Derivatives of Functions

Find the derivatives of the functions in Exercises 1–64.

1. $y = x^5 - 0.125x^2 + 0.25x$

2. $y = 3 - 0.7x^3 + 0.3x^7$

3. $y = x^3 - 3(x^2 + \pi^2)$

4. $y = x^7 + \sqrt{7}x - \dfrac{1}{\pi + 1}$

5. $y = (x + 1)^2(x^2 + 2x)$

6. $y = (2x - 5)(4 - x)^{-1}$

7. $y = (\theta^2 + \sec\theta + 1)^3$

8. $y = \left(-1 - \dfrac{\csc\theta}{2} - \dfrac{\theta^2}{4}\right)^2$

9. $s = \dfrac{\sqrt{t}}{1 + \sqrt{t}}$

10. $s = \dfrac{1}{\sqrt{t} - 1}$

11. $y = 2\tan^2 x - \sec^2 x$

12. $y = \dfrac{1}{\sin^2 x} - \dfrac{2}{\sin x}$

13. $s = \cos^4(1 - 2t)$

14. $s = \cot^3\left(\dfrac{2}{t}\right)$

15. $s = (\sec t + \tan t)^5$

16. $s = \csc^5(1 - t + 3t^2)$

17. $r = \sqrt{2\theta}\sin\theta$

18. $r = 2\theta\sqrt{\cos\theta}$

19. $r = \sin\sqrt{2\theta}$

20. $r = \sin\left(\theta + \sqrt{\theta + 1}\right)$

21. $y = \dfrac{1}{2}x^2\csc\dfrac{2}{x}$

22. $y = 2\sqrt{x}\sin\sqrt{x}$

23. $y = x^{-1/2}\sec(2x)^2$

24. $y = \sqrt{x}\csc(x + 1)^3$

25. $y = 5\cot x^2$

26. $y = x^2\cot 5x$

27. $y = x^2\sin^2(2x^2)$

28. $y = x^{-2}\sin^2(x^3)$

29. $s = \left(\dfrac{4t}{t + 1}\right)^{-2}$

30. $s = \dfrac{-1}{15(15t - 1)^3}$

31. $y = \left(\dfrac{\sqrt{x}}{1 + x}\right)^2$

32. $y = \left(\dfrac{2\sqrt{x}}{2\sqrt{x} + 1}\right)^2$

33. $y = \sqrt{\dfrac{x^2 + x}{x^2}}$

34. $y = 4x\sqrt{x + \sqrt{x}}$

35. $r = \left(\dfrac{\sin\theta}{\cos\theta - 1}\right)^2$

36. $r = \left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

37. $y = (2x + 1)\sqrt{2x + 1}$

38. $y = 20(3x - 4)^{1/4}(3x - 4)^{-1/5}$

39. $y = \dfrac{3}{(5x^2 + \sin 2x)^{3/2}}$

40. $y = (3 + \cos^3 3x)^{-1/3}$

41. $y = 10e^{-x/5}$

42. $y = \sqrt{2}e^{\sqrt{2}x}$

43. $y = \dfrac{1}{4}xe^{4x} - \dfrac{1}{16}e^{4x}$

44. $y = x^2e^{-2/x}$

45. $y = \ln(\sin^2\theta)$

46. $y = \ln(\sec^2\theta)$

47. $y = \log_2(x^2/2)$

48. $y = \log_5(3x - 7)$

49. $y = 8^{-t}$

50. $y = 9^{2t}$

51. $y = 5x^{3.6}$

52. $y = \sqrt{2}x^{-\sqrt{2}}$

53. $y = (x + 2)^{x+2}$

54. $y = 2(\ln x)^{x/2}$

55. $y = \sin^{-1}\sqrt{1 - u^2}, \quad 0 < u < 1$

56. $y = \sin^{-1}\left(\dfrac{1}{\sqrt{v}}\right), \quad v > 1$

57. $y = \ln \cos^{-1} x$

58. $y = z \cos^{-1} z - \sqrt{1 - z^2}$

59. $y = t \tan^{-1} t - \frac{1}{2} \ln t$

60. $y = (1 + t^2) \cot^{-1} 2t$

61. $y = z \sec^{-1} z - \sqrt{z^2 - 1}, \quad z > 1$

62. $y = 2\sqrt{x - 1} \sec^{-1} \sqrt{x}$

63. $y = \csc^{-1}(\sec \theta), \quad 0 < \theta < \pi/2$

64. $y = (1 + x^2) e^{\tan^{-1} x}$

Implicit Differentiation

In Exercises 65–78, find dy/dx by implicit differentiation.

65. $xy + 2x + 3y = 1$

66. $x^2 + xy + y^2 - 5x = 2$

67. $x^3 + 4xy - 3y^{4/3} = 2x$

68. $5x^{4/5} + 10y^{6/5} = 15$

69. $\sqrt{xy} = 1$

70. $x^2y^2 = 1$

71. $y^2 = \frac{x}{x + 1}$

72. $y^2 = \sqrt{\frac{1 + x}{1 - x}}$

73. $e^{x+2y} = 1$

74. $y^2 = 2e^{-1/x}$

75. $\ln(x/y) = 1$

76. $x \sin^{-1} y = 1 + x^2$

77. $ye^{\tan^{-1} x} = 2$

78. $x^y = \sqrt{2}$

In Exercises 79 and 80, find dp/dq.

79. $p^3 + 4pq - 3q^2 = 2$

80. $q = (5p^2 + 2p)^{-3/2}$

In Exercises 81 and 82, find dr/ds.

81. $r \cos 2s + \sin^2 s = \pi$

82. $2rs - r - s + s^2 = -3$

83. Find d^2y/dx^2 by implicit differentiation:

a. $x^3 + y^3 = 1$ b. $y^2 = 1 - \frac{2}{x}$

84. a. By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.

b. Then show that $d^2y/dx^2 = -1/y^3$.

Numerical Values of Derivatives

85. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	−3	1/2
1	3	5	1/2	−4

Find the first derivatives of the following combinations at the given value of x.

a. $6f(x) - g(x), \quad x = 1$

b. $f(x)g^2(x), \quad x = 0$

c. $\frac{f(x)}{g(x) + 1}, \quad x = 1$

d. $f(g(x)), \quad x = 0$

e. $g(f(x)), \quad x = 0$

f. $(x + f(x))^{3/2}, \quad x = 1$

g. $f(x + g(x)), \quad x = 0$

86. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	−2
1	−3	1/5

Find the first derivatives of the following combinations at the given value of x.

a. $\sqrt{x} f(x), \quad x = 1$

b. $\sqrt{f(x)}, \quad x = 0$

c. $f(\sqrt{x}), \quad x = 1$

d. $f(1 - 5 \tan x), \quad x = 0$

e. $\frac{f(x)}{2 + \cos x}, \quad x = 0$

f. $10 \sin\left(\frac{\pi x}{2}\right) f^2(x), \quad x = 1$

87. Find the value of dy/dt at $t = 0$ if $y = 3 \sin 2x$ and $x = t^2 + \pi$.

88. Find the value of ds/du at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

89. Find the value of dw/ds at $s = 0$ if $w = \sin\left(e^{\sqrt{r}}\right)$ and $r = 3 \sin(s + \pi/6)$.

90. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

91. If $y^3 + y = 2 \cos x$, find the value of d^2y/dx^2 at the point (0, 1).

92. If $x^{1/3} + y^{1/3} = 4$, find d^2y/dx^2 at the point (8, 8).

Applying the Derivative Definition

In Exercises 93 and 94, find the derivative using the definition.

93. $f(t) = \frac{1}{2t + 1}$

94. $g(x) = 2x^2 + 1$

95. a. Graph the function

$$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$$

b. Is f continuous at $x = 0$?

c. Is f differentiable at $x = 0$?

Give reasons for your answers.

96. a. Graph the function

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

b. Is f continuous at $x = 0$?

c. Is f differentiable at $x = 0$?

Give reasons for your answers.

97. a. Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

b. Is f continuous at $x = 1$?

c. Is f differentiable at $x = 1$?

Give reasons for your answers.

98. For what value or values of the constant m, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \leq 0 \\ mx, & x > 0 \end{cases}$$

a. continuous at $x = 0$?

b. differentiable at $x = 0$?

Give reasons for your answers.

Slopes, Tangents, and Normals

99. Tangent lines with specified slope Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.

100. Tangent lines with specified slope Are there any points on the curve $y = x - e^{-x}$ where the slope is 2? If so, find them.

101. Horizontal tangent lines Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent line is parallel to the x-axis.

102. Tangent intercepts Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

103. Tangent lines perpendicular or parallel to lines Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent line is

a. perpendicular to the line $y = 1 - (x/24)$.

b. parallel to the line $y = \sqrt{2} - 12x$.

104. Intersecting tangent lines Show that the tangent lines to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

105. Normal lines parallel to a line Find the points on the curve $y = \tan x, -\pi/2 < x < \pi/2$, where the normal line is parallel to the line $y = -x/2$. Sketch the curve and normal lines together, labeling each with its equation.

106. Tangent lines and normal lines Find equations for the tangent and normal lines to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent line, and normal line together, labeling each with its equation.

107. Tangent parabola The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find C.

108. Slope of a tangent line Show that the tangent line to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3).

109. Tangent curve For what value of c is the curve $y = c/(x + 1)$ tangent to the line through the points $(0, 3)$ and $(5, -2)$?

110. Normal lines to a circle Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

In Exercises 111–116, find equations for the lines that are tangent and normal to the curve at the given point.

111. $x^2 + 2y^2 = 9, \quad (1, 2)$

112. $e^x + y^2 = 2, \quad (0, 1)$

113. $xy + 2x - 5y = 2, \quad (3, 2)$

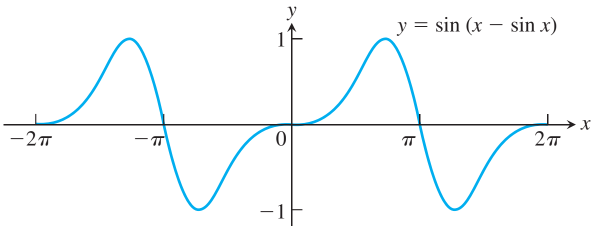
114. $(y - x)^2 = 2x + 4, \quad (6, 2)$

115. $x + \sqrt{xy} = 6, \quad (4, 1)$

116. $x^{3/2} + 2y^{3/2} = 17, \quad (1, 4)$

117. Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points $(1, 1)$ and $(1, -1)$.

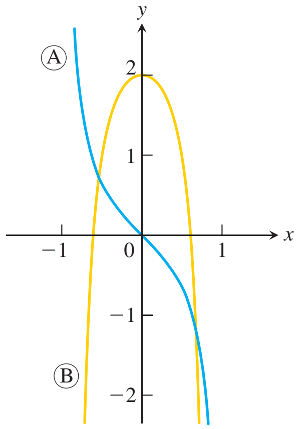
118. The graph shown suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangent lines at the x-axis. Does it? Give reasons for your answer.



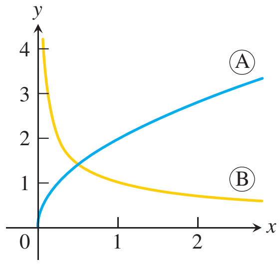
Analyzing Graphs

Each of the figures in Exercises 119 and 120 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?

119.



120.

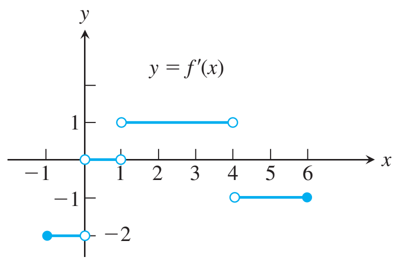


121. Use the following information to graph the function $y = f(x)$ for $-1 \leq x \leq 6$.

i) The graph of f is made of line segments joined end to end.

ii) The graph starts at the point $(-1, 2)$.

iii) The derivative of f, where defined, agrees with the step function shown here.



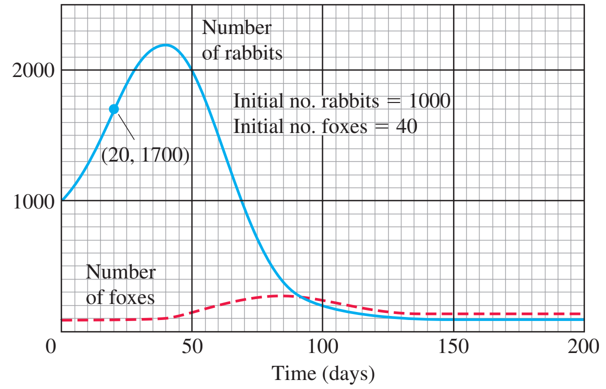
122. Repeat Exercise 121, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 123 and 124 are about the accompanying graphs. The graphs in part (a) show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Part (b) shows the graph of the derivative of the rabbit population, made by plotting slopes.

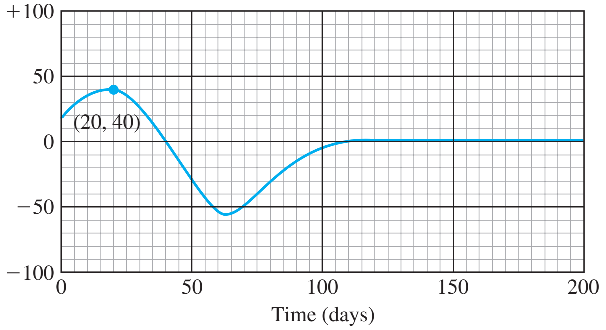
123. a. What is the value of the derivative of the rabbit population when the number of rabbits is largest? Smallest?

b. What is the size of the rabbit population when its derivative is largest? Smallest (negative value)?

124. In what units should the slopes of the rabbit and fox population curves be measured?



(a)



Derivative of the rabbit population

(b)

Source: NCPMF “Differentiation” by W.U. Walton et al., Project CALC. Reprinted by permission of Educational Development Center, Inc.

Trigonometric Limits

Find the limits in Exercises 125–132.

125. $\lim_{x\to 0} \dfrac{\sin x}{2x^2 - x}$

126. $\lim_{x\to 0} \dfrac{3x - \tan 7x}{2x}$

127. $\lim_{r\to 0} \dfrac{\sin r}{\tan 2r}$

128. $\lim_{\theta\to 0} \dfrac{\sin(\sin\theta)}{\theta}$

129. $\lim_{\theta\to(\pi/2)^-} \dfrac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5}$

130. $\lim_{\theta\to 0^+} \dfrac{1 - 2\cot^2\theta}{5\cot^2\theta - 7\cot\theta - 8}$

131. $\lim_{x\to 0} \dfrac{x\sin x}{2 - 2\cos x}$

132. $\lim_{\theta\to 0} \dfrac{1 - \cos\theta}{\theta^2}$

Show how to extend the functions in Exercises 133 and 134 to be continuous at the origin.

133. $g(x) = \dfrac{\tan(\tan x)}{\tan x}$

134. $f(x) = \dfrac{\tan(\tan x)}{\sin(\sin x)}$

Logarithmic Differentiation

In Exercises 135–140, use logarithmic differentiation to find the derivative of y with respect to the appropriate variable.

135. $y = \dfrac{2(x^2 + 1)}{\sqrt{\cos 2x}}$

136. $y = \sqrt[10]{\dfrac{3x + 4}{2x - 4}}$

137. $y = \left(\dfrac{(t + 1)(t - 1)}{(t - 2)(t + 3)}\right)^5, \quad t > 2$

138. $y = \dfrac{2u2^u}{\sqrt{u^2 + 1}}$

139. $y = (\sin\theta)^{\sqrt{\theta}}$

140. $y = (\ln x)^{1/(\ln x)}$

Related Rates

141. Right circular cylinder The total surface area S of a right circular cylinder is related to the base radius r and height h by the equation $S = 2\pi r^2 + 2\pi rh$.

a. How is dS/dt related to dr/dt if h is constant?

b. How is dS/dt related to dh/dt if r is constant?

c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

d. How is dr/dt related to dh/dt if S is constant?

142. Right circular cone The lateral surface area S of a right circular cone is related to the base radius r and height h by the equation $S = \pi r\sqrt{r^2 + h^2}$.

a. How is dS/dt related to dr/dt if h is constant?

b. How is dS/dt related to dh/dt if r is constant?

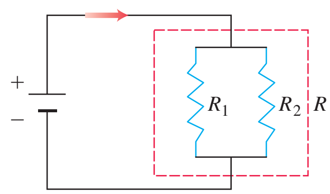
c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

143. Circle's changing area The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

144. Cube's changing edges The volume of a cube is increasing at the rate of 1200 cm^3/min at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

145. Resistors connected in parallel If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$



If R_1 is decreasing at the rate of 1 ohm/sec and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

146. Impedance in a series circuit The impedance Z (ohms) in a series circuit is related to the resistance R (ohms) and reactance X (ohms) by the equation $Z = \sqrt{R^2 + X^2}$. If R is increasing at 3 ohms/sec and X is decreasing at 2 ohms/sec, at what rate is Z changing when $R = 10$ ohms and $X = 20$ ohms?

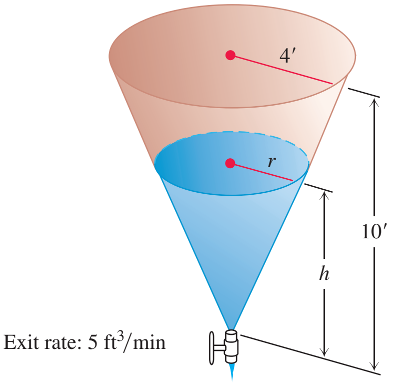
147. Speed of moving particle The coordinates of a particle moving in the metric xy-plane are differentiable functions of time t with $dx/dt = 10$ m/sec and $dy/dt = 5$ m/sec. How fast is the particle moving away from the origin as it passes through the point $(3, -4)$?

148. Motion of a particle A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find dx/dt when $x = 3$.

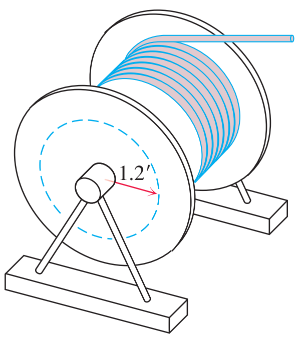
149. Draining a tank Water drains from the conical tank shown in the accompanying figure at the rate of $5\ \text{ft}^3/\text{min}$.

a. What is the relation between the variables h and r in the figure?

b. How fast is the water level dropping when $h = 6$ ft?



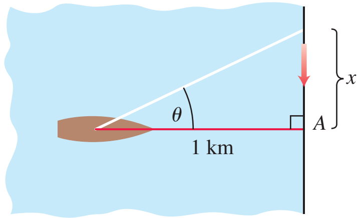
150. Rotating spool As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see accompanying figure). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (radians per second) the spool is turning when the layer of radius 1.2 ft is being unwound.



151. Moving searchlight beam The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

a. How fast is the light moving along the shore when it reaches point A?

b. How many revolutions per minute is 0.6 rad/sec?



152. Points moving on coordinate axes Points A and B move along the x- and y-axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to the line AB remains constant. How fast is OA changing, and is it increasing, or decreasing, when $OB = 2r$ and B is moving toward O at the rate of $0.3r$ m/sec?

Linearization

153. Find the linearizations of

a. $\tan x$ at $x = -\pi/4$ **b.** $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

154. We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1 - x \quad \text{and} \quad \tan x \approx x$$

to get

$$\frac{1}{1+\tan x} \approx 1 - x.$$

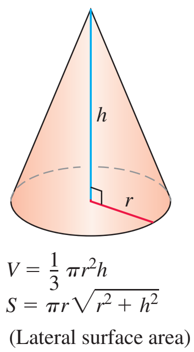
Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

155. Find the linearization of $f(x) = \sqrt{1+x} + \sin x - 0.5$ at $x = 0$.

156. Find the linearization of $f(x) = 2/(1 - x) + \sqrt{1+x} - 3.1$ at $x = 0$.

Differential Estimates of Change

157. Surface area of a cone Write a formula that estimates the change that occurs in the lateral surface area of a right circular cone when the height changes from h_0 to $h_0 + dh$ and the radius does not change.



158. Controlling error

a. How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?

b. Suppose that the edge is measured with the accuracy required in part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement.

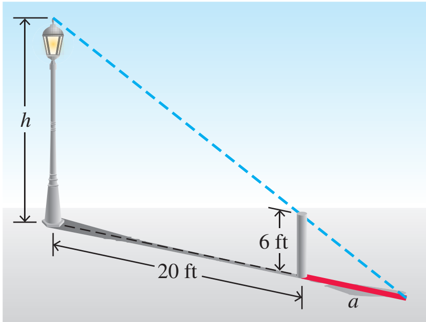
159. Compounding error The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of

a. the radius.

b. the surface area.

c. the volume.

160. Finding height To find the height of a lamppost (see accompanying figure), you stand a 6 ft pole 20 ft from the lamp and measure the length a of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$ and estimate the possible error in the result.



CHAPTER 3 Additional and Advanced Exercises

1. An equation like $\sin^2\theta + \cos^2\theta = 1$ is called an **identity** because it holds for all values of θ. An equation like $\sin\theta = 0.5$ is not an identity because it holds only for selected values of θ, not all. If you differentiate both sides of a trigonometric identity in θ with respect to θ, the resulting new equation will also be an identity.

Differentiate the following to show that the resulting equations hold for all θ.

a. $\sin 2\theta = 2\sin\theta\cos\theta$

b. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

2. If the identity $\sin(x + a) = \sin x\cos a + \cos x\sin a$ is differentiated with respect to x, is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.

3. a. Find values for the constants a, b, and c that will make

$$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

satisfy the conditions

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

b. Find values for b and c that will make

$$f(x) = \sin(x + a) \quad \text{and} \quad g(x) = b\sin x + c\cos x$$

satisfy the conditions

$$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

c. For the determined values of a, b, and c, what happens for the third and fourth derivatives of f and g in each of parts (a) and (b)?

4. Solutions to differential equations

a. Show that $y = \sin x$, $y = \cos x$, and $y = a\ \cos x + b\sin x$ (a and b constants) all satisfy the equation

$$y'' + y = 0.$$

b. How would you modify the functions in part (a) to satisfy the equation

$$y'' + 4y = 0?$$

Generalize this result.

5. An osculating circle Find the values of h, k, and a that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point (1, 2) and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari*, meaning "to kiss"). We encounter them again in Chapter 13.

6. Marginal revenue A bus will hold 60 people. The number x of people per trip who use the bus is related to the fare charged (p dollars) by the law $p = [3 - (x/40)]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue dr/dx equal to zero? What is the corresponding fare? (This fare is the one that maximizes the revenue.)

7. Industrial production

a. Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.