Proof of Law 1 Let

$$y_1 = e^{x_1} \quad \text{and} \quad y_2 = e^{x_2}. \tag{4}$$

Then

$$
\begin{aligned}
x_1 &= \ln y_1 \quad \text{and} \quad x_2 = \ln y_2 && \text{Take logs of both sides of Eqs. (4).}\\
x_1 + x_2 &= \ln y_1 + \ln y_2 \\
&= \ln y_1 y_2 && \text{Product Rule for logarithms}\\
e^{x_1+x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.}\\
&= y_1 y_2 && e^{\ln u} = u\\
&= e^{x_1}e^{x_2}.
\end{aligned}
$$

■

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercises 77 and 78).

EXERCISES 4.2

Checking the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–8.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$

2. $f(x) = x^{2/3}, \quad [0, 1]$

3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$

4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

5. $f(x) = \sin^{-1} x, \quad [-1, 1]$

6. $f(x) = \ln (x - 1), \quad [2, 4]$

7. $f(x) = x^3 - x^2, \quad [-1, 2]$

8. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Which of the functions in Exercises 9–14 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

9. $f(x) = x^{2/3}, \quad [-1, 8]$

10. $f(x) = x^{4/5}, \quad [0, 1]$

11. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$

12. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$

13. $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$

14. $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$

15. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

16. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Roots (Zeros)

17. a. Plot the zeros of each polynomial on a line together with the zeros of its first derivative.

i) $y = x^2 - 4$

ii) $y = x^2 + 8x + 15$

iii) $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$

iv) $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$

b. Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ there lies a zero of

$$nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + a_1.$$

18. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.

19. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?

20. Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 21–28 have exactly one zero in the given interval.

21. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$

22. $f(x) = x^3 + \frac{4}{x^2} + 7, \quad (-\infty, 0)$

23. $g(t) = \sqrt{t} + \sqrt{1 + t} - 4, \quad (0, \infty)$

24. $g(t) = \dfrac{1}{1 - t} + \sqrt{1 + t} - 3.1, \quad (-1, 1)$

25. $r(\theta) = \theta + \sin^2\left(\dfrac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$

26. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$

27. $r(\theta) = \sec\theta - \dfrac{1}{\theta^3} + 5, \quad (0, \pi/2)$

28. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

Finding Functions from Derivatives

29. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.

30. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

31. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a. $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.

32. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 33–38, find all possible functions with the given derivative.

33. a. $y' = x$ **b.** $y' = x^2$ **c.** $y' = x^3$

34. a. $y' = 2x$ **b.** $y' = 2x - 1$ **c.** $y' = 3x^2 + 2x - 1$

35. a. $y' = -\dfrac{1}{x^2}$ **b.** $y' = 1 - \dfrac{1}{x^2}$ **c.** $y' = 5 + \dfrac{1}{x^2}$

36. a. $y' = \dfrac{1}{2\sqrt{x}}$ **b.** $y' = \dfrac{1}{\sqrt{x}}$ **c.** $y' = 4x - \dfrac{1}{\sqrt{x}}$

37. a. $y' = \sin 2t$ **b.** $y' = \cos\dfrac{t}{2}$ **c.** $y' = \sin 2t + \cos\dfrac{t}{2}$

38. a. $y' = \sec^2\theta$ **b.** $y' = \sqrt{\theta}$ **c.** $y' = \sqrt{\theta} - \sec^2\theta$

In Exercises 39–42, find the function with the given derivative whose graph passes through the point P.

39. $f'(x) = 2x - 1, \quad P(0, 0)$

40. $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

41. $f'(x) = e^{2x}, \quad P\left(0, \dfrac{3}{2}\right)$

42. $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

Finding Position from Velocity or Acceleration

Exercises 43–46 give the velocity $v = ds/dt$ and initial position of an object moving along a coordinate line. Find the object's position at time t.

43. $v = 9.8t + 5, \quad s(0) = 10$ **44.** $v = 32t - 2, \quad s(0.5) = 4$

45. $v = \sin \pi t, \quad s(0) = 0$ **46.** $v = \dfrac{2}{\pi}\cos\dfrac{2t}{\pi}, \quad s(\pi^2) = 1$

Exercises 47–50 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of an object moving on a coordinate line. Find the object's position at time t.

47. $a = e^t, \quad v(0) = 20, \quad s(0) = 5$

48. $a = 9.8, \quad v(0) = -3, \quad s(0) = 0$

49. $a = -4\sin 2t, \quad v(0) = 2, \quad s(0) = -3$

50. $a = \dfrac{9}{\pi^2}\cos\dfrac{3t}{\pi}, \quad v(0) = 0, \quad s(0) = -1$

Applications

51. Temperature change It took 14 sec for a mercury thermometer to rise from $-19°C$ to $100°C$ when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at the rate of $8.5°C/\text{sec}$.

52. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

53. Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea or nautical miles per hour).

54. A marathoner ran the 26.2-mi New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph, assuming the initial and final speeds are zero.

55. Show that at some instant during a 2-hour automobile trip the car's speedometer reading will equal the average speed for the trip.

56. Free fall on the moon On our moon, the acceleration of gravity is $1.6\ \text{m/sec}^2$. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

Theory and Examples

57. The geometric mean of *a* and *b* The *geometric mean* of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

58. The arithmetic mean of *a* and *b* The *arithmetic mean* of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T **59.** Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

60. Rolle's Theorem

a. Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and 2.

b. Graph f and its derivative f' together. How is what you see related to Rolle's Theorem?

c. Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon as f and f'?

61. Unique solution Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

62. Parallel tangents Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

63. Suppose that $f'(x) \leq 1$ for $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

64. Suppose that $0 < f'(x) < 1/2$ for all x-values. Show that $f(-1) < f(1) < 2 + f(-1)$.

65. Show that $|\cos x - 1| \le |x|$ for all x-values. (*Hint:* Consider $f(t) = \cos t$ on $[0, x]$.)

66. Show that for any numbers a and b, the sine inequality $|\sin b - \sin a| \le |b - a|$ is true.

67. If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

68. If $|f(w) - f(x)| \le |w - x|$ for all values w and x and f is a differentiable function, show that $-1 \le f'(x) \le 1$ for all x-values.

69. Assume that f is differentiable on $a \le x \le b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

70. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \le \frac{f(b) - f(a)}{b - a} \le \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answers.

T **71.** Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \le x \le 0.1$ and $f(0) = 1$.

T **72.** Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \le x \le 0.1$ and $f(0) = 2$.

73. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

a. Show that $f(x) \ge 1$ for all x.

b. Must $f'(1) = 0$? Explain.

74. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.

75. Use the same-derivative argument, as was done to prove the Product and Power Rules for logarithms, to prove the Quotient Rule property.

76. Use the same-derivative argument to prove the identities

a. $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$ **b.** $\sec^{-1} x + \csc^{-1} x = \dfrac{\pi}{2}$

77. Starting with the equation $e^{x_1}e^{x_2} = e^{x_1 + x_2}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number x. Then show that $e^{x_1}/e^{x_2} = e^{x_1 - x_2}$ for any numbers x_1 and x_2.

78. Show that $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$ for any numbers x_1 and x_2.

4.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function, it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

Increasing Functions and Decreasing Functions

As another corollary to the Mean Value Theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

> **COROLLARY 3** Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
>
> If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
> If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b) and $f(x_2) < f(x_1)$ if f' is negative on (a, b). ■