EXERCISES 4.3

Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of f?

b. On what open intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$

2. $f'(x) = (x - 1)(x + 2)$

3. $f'(x) = (x - 1)^2(x + 2)$

4. $f'(x) = (x - 1)^2(x + 2)^2$

5. $f'(x) = (x - 1)e^{-x}$

6. $f'(x) = (x - 7)(x + 1)(x + 5)$

7. $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$

8. $f'(x) = \dfrac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, \quad x \neq -1, 3$

9. $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$

10. $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$

11. $f'(x) = x^{-1/3}(x + 2)$

12. $f'(x) = x^{-1/2}(x - 3)$

13. $f'(x) = (\sin x - 1)(2\cos x + 1), \; 0 \le x \le 2\pi$

14. $f'(x) = (\sin x + \cos x)(\sin x - \cos x), \; 0 \le x \le 2\pi$

Identifying Extrema

In Exercises 15–46:

a. Find the open intervals on which the function is increasing and decreasing.

b. Identify the function's local and absolute extreme values, if any, saying where they occur.

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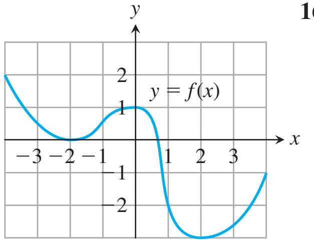

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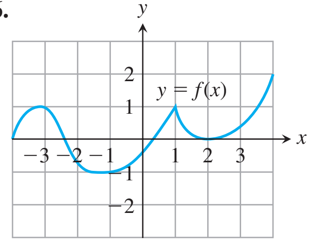

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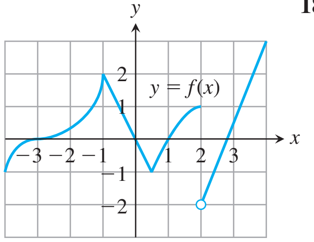

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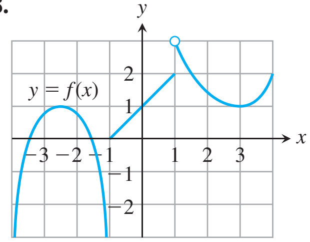


19. $g(t) = -t^2 - 3t + 3$

20. $g(t) = -3t^2 + 9t + 5$

21. $h(x) = -x^3 + 2x^2$

22. $h(x) = 2x^3 - 18x$

23. $f(\theta) = 3\theta^2 - 4\theta^3$

24. $f(\theta) = 6\theta - \theta^3$

25. $f(r) = 3r^3 + 16r$

26. $h(r) = (r + 7)^3$

27. $f(x) = x^4 - 8x^2 + 16$

28. $g(x) = x^4 - 4x^3 + 4x^2$

29. $H(t) = \dfrac{3}{2}t^4 - t^6$

30. $K(t) = 15t^3 - t^5$

31. $f(x) = x - 6\sqrt{x - 1}$

32. $g(x) = 4\sqrt{x} - x^2 + 3$

33. $g(x) = x\sqrt{8 - x^2}$

34. $g(x) = x^2\sqrt{5 - x}$

35. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$

36. $f(x) = \dfrac{x^3}{3x^2 + 1}$

37. $f(x) = x^{1/3}(x + 8)$

38. $g(x) = x^{2/3}(x + 5)$

39. $h(x) = x^{1/3}(x^2 - 4)$

40. $k(x) = x^{2/3}(x^2 - 4)$

41. $f(x) = e^{2x} + e^{-x}$

42. $f(x) = e^{\sqrt{x}}$

43. $f(x) = x \ln x$

44. $f(x) = x^2 \ln x$

45. $g(x) = x(\ln x)^2$

46. $g(x) = x^2 - 2x - 4\ln x$

In Exercises 47–58:

a. Identify the function's local extreme values in the given domain, and say where they occur.

b. Which of the extreme values, if any, are absolute?

T **c.** Support your findings with a graphing calculator or computer grapher.

47. $f(x) = 2x - x^2, \quad -\infty < x \le 2$

48. $f(x) = (x + 1)^2, \quad -\infty < x \le 0$

49. $g(x) = x^2 - 4x + 4, \quad 1 \le x < \infty$

50. $g(x) = -x^2 - 6x - 9, \quad -4 \le x < \infty$

51. $f(t) = 12t - t^3, \quad -3 \le t < \infty$

52. $f(t) = t^3 - 3t^2, \quad -\infty < t \le 3$

53. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \le x < \infty$

54. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \le 0$

55. $f(x) = \sqrt{25 - x^2}, \quad -5 \le x \le 5$

56. $f(x) = \sqrt{x^2 - 2x - 3}, \quad 3 \le x < \infty$

57. $g(x) = \dfrac{x - 2}{x^2 - 1}, \quad 0 \le x < 1$

58. $g(x) = \dfrac{x^2}{4 - x^2}, \quad -2 < x \le 1$

In Exercises 59–66:

a. Find the local extrema of each function on the given interval, and say where they occur.

T **b.** Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f'.

59. $f(x) = \sin 2x, \quad 0 \le x \le \pi$

60. $f(x) = \sin x - \cos x, \quad 0 \le x \le 2\pi$

61. $f(x) = \sqrt{3}\cos x + \sin x, \quad 0 \le x \le 2\pi$

62. $f(x) = -2x + \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

63. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \le x \le 2\pi$

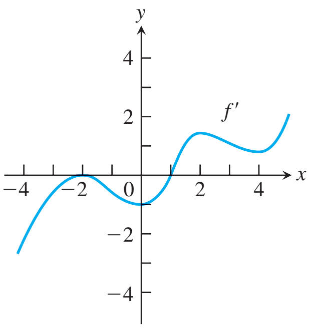
64. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \le x \le \pi$

65. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

66. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

In Exercises 67 and 68, the graph of f' is given. Assume that f is continuous and determine the x-values corresponding to local minima and local maxima.

67.



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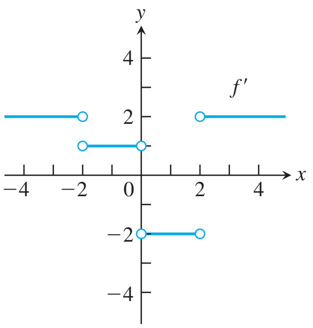


Theory and Examples

Show that the functions in Exercises 69 and 70 have local extreme values at the given values of θ, and say which kind of local extreme the function has.

69. $h(\theta) = 3\cos\dfrac{\theta}{2}, \quad 0 \le \theta \le 2\pi, \quad \text{at } \theta = 0 \text{ and } \theta = 2\pi$

70. $h(\theta) = 5\sin\dfrac{\theta}{2}, \quad 0 \le \theta \le \pi, \quad \text{at } \theta = 0 \text{ and } \theta = \pi$

71. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

a. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

b. $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

c. $f'(x) > 0$ for $x \ne 1$;

d. $f'(x) < 0$ for $x \ne 1$.

72. Sketch the graph of a differentiable function $y = f(x)$ that has

a. a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;

b. a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;

c. local maxima at $(1, 1)$ and $(3, 3)$;

d. local minima at $(1, 1)$ and $(3, 3)$.

73. Sketch the graph of a continuous function $y = g(x)$ such that

a. $g(2) = 2, 0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

b. $g(2) = 2$, $g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

74. Sketch the graph of a continuous function $y = h(x)$ such that

a. $h(0) = 0, -2 \le h(x) \le 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

b. $h(0) = 0, -2 \le h(x) \le 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

75. Discuss the extreme-value behavior of the function $f(x) = x\sin(1/x)$, $x \ne 0$. How many critical points does this function have? Where are they located on the x-axis? Does f have an absolute minimum? An absolute maximum? (See Exercise 49 in Section 2.3.)

76. Find the open intervals on which the function $f(x) = ax^2 + bx + c$, $a \ne 0$, is increasing and decreasing. Describe the reasoning behind your answer.

77. Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

78. Determine the values of constants a, b, c, and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

79. Locate and identify the absolute extreme values of

a. $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

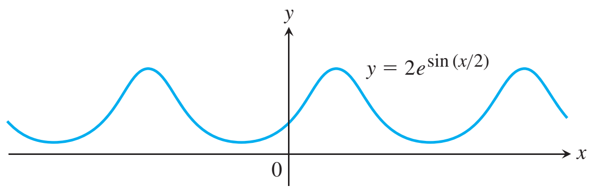
b. $\cos(\ln x)$ on $[1/2, 2]$.

80. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b. Using part (a), show that $\ln x < x$ if $x > 1$.

81. Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

82. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



83. Find the absolute maximum value of $f(x) = x^2\ln(1/x)$ and say where it occurs.

84. a. Prove that $e^x \ge 1 + x$ if $x \ge 0$.

b. Use the result in part (a) to show that

$$e^x \ge 1 + x + \frac{1}{2}x^2.$$

85. Show that increasing functions and decreasing functions are one-to-one. That is, show that for any x_1 and x_2 in I, $x_2 \ne x_1$ implies $f(x_2) \ne f(x_1)$.

Use the results of Exercise 85 to show that the functions in Exercises 86–90 have inverses over their domains. Find a formula for df^{-1}/dx using Theorem 3, Section 3.8.

86. $f(x) = (1/3)x + (5/6)$

87. $f(x) = 27x^3$

88. $f(x) = 1 - 8x^3$

89. $f(x) = (1 - x)^3$

90. $f(x) = x^{5/3}$

4.4 Concavity and Curve Sketching

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic