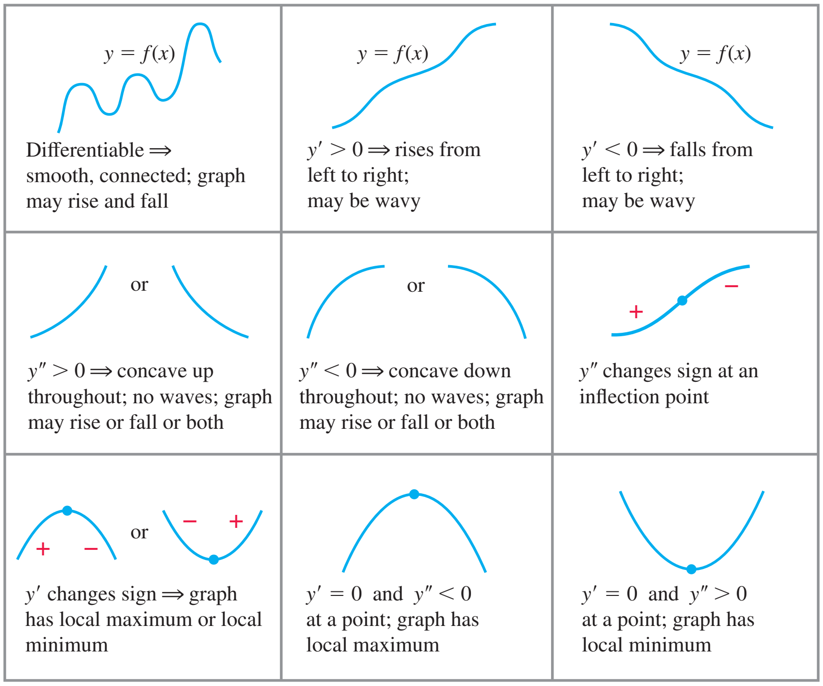
asymptotes is found using limits (Section 2.6). The following figure summarizes how the first derivative and second derivative affect the shape of a graph.

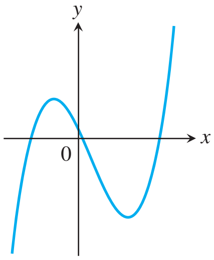


EXERCISES 4.4

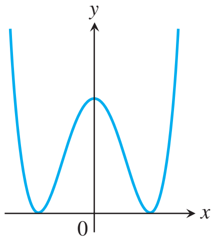
Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

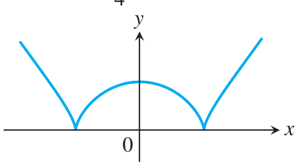
1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



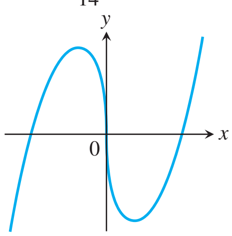
2. $y = \frac{x^4}{4} - 2x^2 + 4$



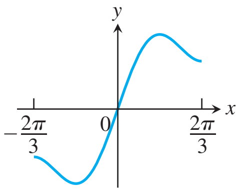
3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



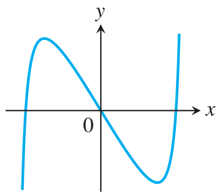
4. $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



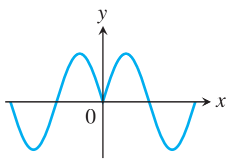
5. $y = x + \sin 2x,\ -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



6. $y = \tan x - 4x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

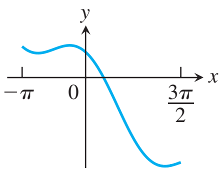


7. $y = \sin|x|,\ -2\pi \le x \le 2\pi$



NOT TO SCALE

8. $y = 2\cos x - \sqrt{2}x,\ -\pi \le x \le \frac{3\pi}{2}$



Graphing Functions

In Exercises 9–58, identify the coordinates of any local and absolute extreme points and inflection points. Graph the function.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$ **20.** $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$ **22.** $y = x\left(\frac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

26. $y = \frac{4}{3}x - \tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

27. $y = \sin x \cos x, \quad 0 \le x \le \pi$

28. $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

29. $y = x^{1/5}$ **30.** $y = x^{2/5}$

31. $y = \frac{x}{\sqrt{x^2 + 1}}$ **32.** $y = \frac{\sqrt{1 - x^2}}{2x + 1}$

33. $y = 2x - 3x^{2/3}$ **34.** $y = 5x^{2/5} - 2x$

35. $y = x^{2/3}\left(\frac{5}{2} - x\right)$ **36.** $y = x^{2/3}(x - 5)$

37. $y = x\sqrt{8 - x^2}$ **38.** $y = (2 - x^2)^{3/2}$

39. $y = \sqrt{16 - x^2}$ **40.** $y = x^2 + \frac{2}{x}$

41. $y = \frac{x^2 - 3}{x - 2}$ **42.** $y = \sqrt[3]{x^3 + 1}$

43. $y = \frac{8x}{x^2 + 4}$ **44.** $y = \frac{5}{x^4 + 5}$

45. $y = |x^2 - 1|$ **46.** $y = |x^2 - 2x|$

47. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

48. $y = \sqrt{|x - 4|}$

49. $y = \frac{x}{9 - x^2}$ **50.** $y = \frac{x^2}{1 - x}$

51. $y = \ln(3 - x^2)$ **52.** $y = (\ln x)^2$

53. $y = e^x - 2e^{-x} - 3x$ **54.** $y = xe^{-x}$

55. $y = \ln(\cos x)$ **56.** $y = \frac{\ln x}{\sqrt{x}}$

57. $y = \frac{1}{1 + e^{-x}}$ **58.** $y = \frac{e^x}{1 + e^x}$

Sketching the General Shape, Knowing y'

Each of Exercises 59–80 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use Steps 2–4 of the graphing procedure on page 249 to sketch the general shape of the graph of f.

59. $y' = 2 + x - x^2$ **60.** $y' = x^2 - x - 6$

61. $y' = x(x - 3)^2$ **62.** $y' = x^2(2 - x)$

63. $y' = x(x^2 - 12)$ **64.** $y' = (x - 1)^2(2x + 3)$

65. $y' = (8x - 5x^2)(4 - x)^2$ **66.** $y' = (x^2 - 2x)(x - 5)^2$

67. $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

68. $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

69. $y' = \cot\frac{\theta}{2}, \quad 0 < \theta < 2\pi$ **70.** $y' = \csc^2\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

71. $y' = \tan^2\theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

72. $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

73. $y' = \cos t, \quad 0 \le t \le 2\pi$

74. $y' = \sin t, \quad 0 \le t \le 2\pi$

75. $y' = (x + 1)^{-2/3}$ **76.** $y' = (x - 2)^{-1/3}$

77. $y' = x^{-2/3}(x - 1)$ **78.** $y' = x^{-4/5}(x + 1)$

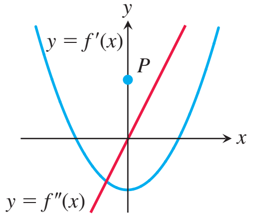
79. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

80. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

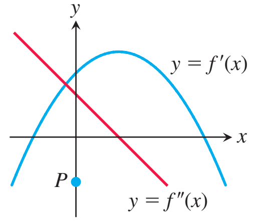
Sketching y from Graphs of y' and y''

Each of Exercises 81–84 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

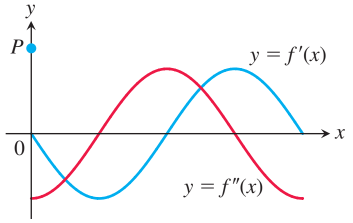
81.



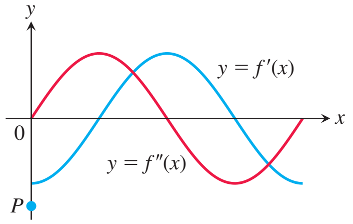
82.



83.



84.



Graphing Rational Functions

Graph the rational functions in Exercises 85–102 using all the steps in the graphing procedure on page 249.

85. $y = \frac{2x^2 + x - 1}{x^2 - 1}$ **86.** $y = \frac{x^2 - 49}{x^2 + 5x - 14}$

87. $y = \frac{x^4 + 1}{x^2}$ **88.** $y = \frac{x^2 - 4}{2x}$

89. $y = \frac{1}{x^2 - 1}$ **90.** $y = \frac{x^2}{x^2 - 1}$

91. $y = -\frac{x^2 - 2}{x^2 - 1}$ **92.** $y = \frac{x^2 - 4}{x^2 - 2}$

93. $y = \frac{x^2}{x + 1}$ **94.** $y = -\frac{x^2 - 4}{x + 1}$

95. $y = \frac{x^2 - x + 1}{x - 1}$ **96.** $y = -\frac{x^2 - x + 1}{x - 1}$

97. $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

98. $y = \dfrac{x^3 + x - 2}{x - x^2}$

99. $y = \dfrac{x}{x^2 - 1}$

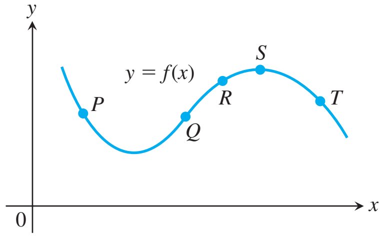
100. $y = \dfrac{x - 1}{x^2(x - 2)}$

101. $y = \dfrac{8}{x^2 + 4}$ (Agnesi's witch)

102. $y = \dfrac{4x}{x^2 + 4}$ (Newton's serpentine)

Theory and Examples

103. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



104. Sketch a smooth connected curve $y = f(x)$ with

$f(-2) = 8,$ $\quad f'(2) = f'(-2) = 0,$

$f(0) = 4,$ $\quad f'(x) < 0$ for $|x| < 2,$

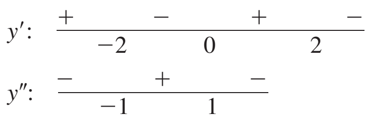
$f(2) = 0,$ $\quad f''(x) < 0$ for $x < 0,$

$f'(x) > 0$ for $|x| > 2,$ $\quad f''(x) > 0$ for $x > 0.$

105. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Derivatives
$x < 2$		$y' < 0, \quad y'' > 0$
2	1	$y' = 0, \quad y'' > 0$
$2 < x < 4$		$y' > 0, \quad y'' > 0$
4	4	$y' > 0, \quad y'' = 0$
$4 < x < 6$		$y' > 0, \quad y'' < 0$
6	7	$y' = 0, \quad y'' < 0$
$x > 6$		$y' < 0, \quad y'' < 0$

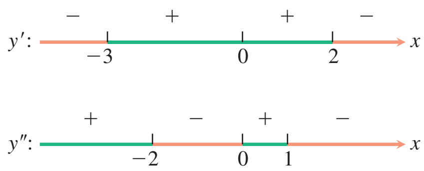
106. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.



107. Sketch the graph of a twice-differentiable $y = f(x)$ with the following properties. Label coordinates where possible.

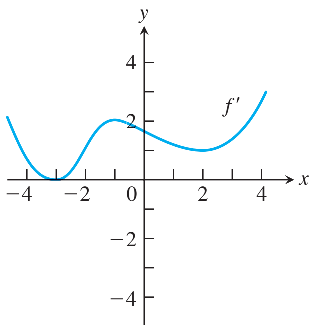
x	y	Derivatives
$x < -2$		$y' > 0, \quad y'' < 0$
-2	-1	$y' = 0, \quad y'' = 0$
$-2 < x < -1$		$y' > 0, \quad y'' > 0$
-1	0	$y' > 0, \quad y'' = 0$
$-1 < x < 0$		$y' > 0, \quad y'' < 0$
0	3	$y' = 0, \quad y'' < 0$
$0 < x < 1$		$y' < 0, \quad y'' < 0$
1	2	$y' < 0, \quad y'' = 0$
$1 < x < 2$		$y' < 0, \quad y'' > 0$
2	0	$y' = 0, \quad y'' > 0$
$x > 2$		$y' > 0, \quad y'' > 0$

108. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-3, -2)$, $(-2, 0)$, $(0, 1)$, $(1, 2)$, and $(2, 3)$ and whose first two derivatives have the following sign patterns.

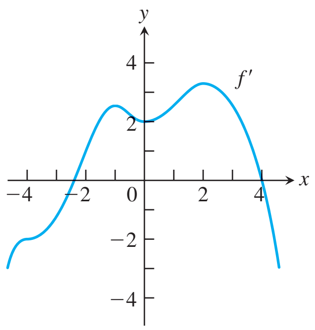


In Exercises 109 and 110, the graph of f' is given. Determine x-values corresponding to inflection points for the graph of f.

109.

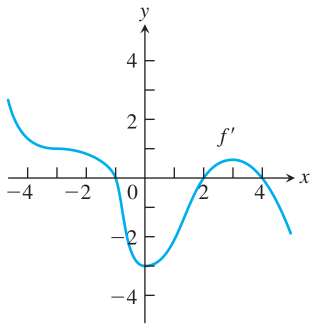


110.

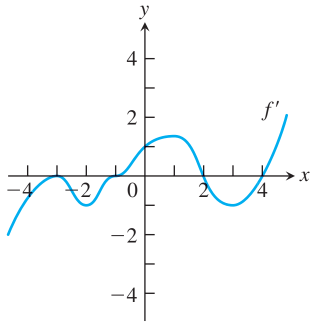


In Exercises 111 and 112, the graph of f' is given. Determine x-values corresponding to local minima, local maxima, and inflection points for the graph of f.

111.

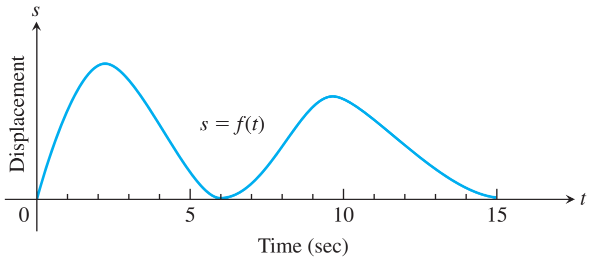


112.

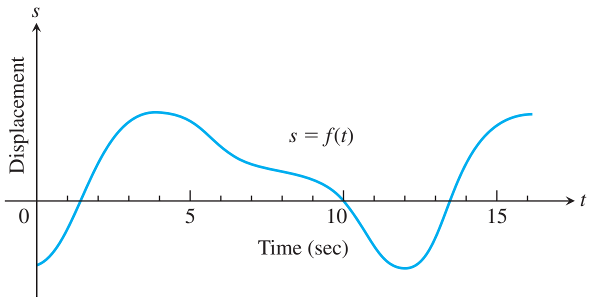


Motion Along a Line The graphs in Exercises 113 and 114 show the position $s = f(t)$ of an object moving up and down on a coordinate line. **(a)** When is the object moving away from the origin? Toward the origin? At approximately what times is the **(b)** velocity equal to zero? **(c)** Acceleration equal to zero? **(d)** When is the acceleration positive? Negative?

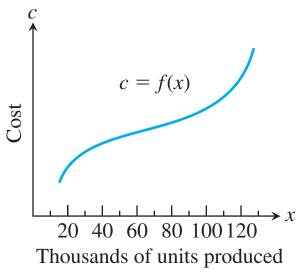
113.



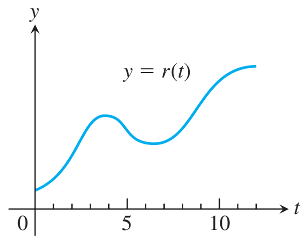
114.



115. Marginal cost The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



116. The accompanying graph shows the monthly revenue of the Widget Corporation for the past 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?



117. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x-1)^2(x-2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for y'.)

118. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x-1)^2(x-2)(x-4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

119. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

120. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

121. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

122. Parabolas

a. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

b. When is the parabola concave up? Concave down? Give reasons for your answers.

123. Quadratic curves What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

124. Cubic curves What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

125. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x + 1)(x - 2).$$

For what x-values does the graph of f have an inflection point?

126. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x - 2)^3(x + 3).$$

For what x-values does the graph of f have an inflection point?

127. Find the values of constants a, b, and c so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

128. Find the values of constants a, b, and c so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

COMPUTER EXPLORATIONS

In Exercises 129–132, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

129. $y = x^5 - 5x^4 - 240$

130. $y = x^3 - 12x^2$

131. $y = \frac{4}{5}x^5 + 16x^2 - 25$

132. $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

133. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

134. Graph $f(x) = x \cos x$ and its second derivative together for $0 \le x \le 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

4.5 Indeterminate Forms and L'Hôpital's Rule

Expressions such as "0/0" and "∞/∞" look something like ordinary numbers. We say that they have the *form* of a number. But values cannot be assigned to them in a way that is consistent with the usual rules to add and multiply numbers. We are led to call them "indeterminate forms." Although we must remain careful to remember that they are not numbers, we will see that they can play useful roles in summarizing the limiting behavior of a function.

HISTORICAL BIOGRAPHY
Guillaume François Antoine de l'Hôpital (1661–1704)
www.goo.gl/nMJIKA
Johann Bernoulli (1667–1748)
www.goo.gl/70BgHS

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of the rule.

Indeterminate Form 0/0

If we want to know how the function

$$f(x) = \frac{3x - \sin x}{x}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $f(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $f(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression that