EXERCISES 4.5

Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x \to -2} \frac{x + 2}{x^2 - 4}$
2. $\lim_{x \to 0} \frac{\sin 5x}{x}$
3. $\lim_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1}$
4. $\lim_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3}$
5. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$
6. $\lim_{x \to \infty} \frac{2x^2 + 3x}{x^3 + x + 1}$

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$
8. $\lim_{x \to -5} \frac{x^2 - 25}{x + 5}$
9. $\lim_{t \to -3} \frac{t^3 - 4t + 15}{t^2 - t - 12}$
10. $\lim_{t \to -1} \frac{3t^3 + 3}{4t^3 - t + 3}$
11. $\lim_{x \to \infty} \frac{5x^3 - 2x}{7x^3 + 3}$
12. $\lim_{x \to \infty} \frac{x - 8x^2}{12x^2 + 5x}$
13. $\lim_{t \to 0} \frac{\sin t^2}{t}$
14. $\lim_{t \to 0} \frac{\sin 5t}{2t}$
15. $\lim_{x \to 0} \frac{8x^2}{\cos x - 1}$
16. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$
17. $\lim_{\theta \to \pi/2} \frac{2\theta - \pi}{\cos(2\pi - \theta)}$
18. $\lim_{\theta \to -\pi/3} \frac{3\theta + \pi}{\sin(\theta + (\pi/3))}$
19. $\lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{1 + \cos 2\theta}$
20. $\lim_{x \to 1} \frac{x - 1}{\ln x - \sin \pi x}$
21. $\lim_{x \to 0} \frac{x^2}{\ln(\sec x)}$
22. $\lim_{x \to \pi/2} \frac{\ln(\csc x)}{(x - (\pi/2))^2}$
23. $\lim_{t \to 0} \frac{t(1 - \cos t)}{t - \sin t}$
24. $\lim_{t \to 0} \frac{t \sin t}{1 - \cos t}$
25. $\lim_{x \to (\pi/2)^-} \left(x - \frac{\pi}{2}\right) \sec x$
26. $\lim_{x \to (\pi/2)^-} \left(\frac{\pi}{2} - x\right) \tan x$
27. $\lim_{\theta \to 0} \frac{3^{\sin \theta} - 1}{\theta}$
28. $\lim_{\theta \to 0} \frac{(1/2)^{\theta} - 1}{\theta}$
29. $\lim_{x \to 0} \frac{x 2^x}{2^x - 1}$
30. $\lim_{x \to 0} \frac{3^x - 1}{2^x - 1}$
31. $\lim_{x \to \infty} \frac{\ln(x + 1)}{\log_2 x}$
32. $\lim_{x \to \infty} \frac{\log_2 x}{\log_3(x + 3)}$
33. $\lim_{x \to 0^+} \frac{\ln(x^2 + 2x)}{\ln x}$
34. $\lim_{x \to 0^+} \frac{\ln(e^x - 1)}{\ln x}$
35. $\lim_{y \to 0} \frac{\sqrt{5y + 25} - 5}{y}$
36. $\lim_{y \to 0} \frac{\sqrt{ay + a^2} - a}{y}, \quad a > 0$
37. $\lim_{x \to \infty} (\ln 2x - \ln(x + 1))$
38. $\lim_{x \to 0^+} (\ln x - \ln \sin x)$
39. $\lim_{x \to 0^+} \frac{(\ln x)^2}{\ln(\sin x)}$
40. $\lim_{x \to 0^+} \left(\frac{3x + 1}{x} - \frac{1}{\sin x}\right)$
41. $\lim_{x \to 1^+} \left(\frac{1}{x - 1} - \frac{1}{\ln x}\right)$
42. $\lim_{x \to 0^+} (\csc x - \cot x + \cos x)$
43. $\lim_{\theta \to 0} \frac{\cos \theta - 1}{e^{\theta} - \theta - 1}$
44. $\lim_{h \to 0} \frac{e^h - (1 + h)}{h^2}$
45. $\lim_{t \to \infty} \frac{e^t + t^2}{e^t - t}$
46. $\lim_{x \to \infty} x^2 e^{-x}$
47. $\lim_{x \to 0} \frac{x - \sin x}{x \tan x}$
48. $\lim_{x \to 0} \frac{(e^x - 1)^2}{x \sin x}$
49. $\lim_{\theta \to 0} \frac{\theta - \sin \theta \cos \theta}{\tan \theta - \theta}$
50. $\lim_{x \to 0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x}$

Indeterminate Powers and Products

Find the limits in Exercises 51–66.

51. $\lim_{x \to 1^+} x^{1/(1-x)}$
52. $\lim_{x \to 1^+} x^{1/(x-1)}$
53. $\lim_{x \to \infty} (\ln x)^{1/x}$
54. $\lim_{x \to e^+} (\ln x)^{1/(x-e)}$
55. $\lim_{x \to 0^+} x^{-1/\ln x}$
56. $\lim_{x \to \infty} x^{1/\ln x}$
57. $\lim_{x \to \infty} (1 + 2x)^{1/(2 \ln x)}$
58. $\lim_{x \to 0} (e^x + x)^{1/x}$
59. $\lim_{x \to 0^+} x^x$
60. $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$
61. $\lim_{x \to \infty} \left(\frac{x + 2}{x - 1}\right)^x$
62. $\lim_{x \to \infty} \left(\frac{x^2 + 1}{x + 2}\right)^{1/x}$
63. $\lim_{x \to 0^+} x^2 \ln x$
64. $\lim_{x \to 0^+} x(\ln x)^2$
65. $\lim_{x \to 0^+} x \tan\left(\frac{\pi}{2} - x\right)$
66. $\lim_{x \to 0^+} \sin x \cdot \ln x$

Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 67–74. Try it—you just keep on cycling. Find the limits some other way.

67. $\lim_{x \to \infty} \frac{\sqrt{9x + 1}}{\sqrt{x + 1}}$
68. $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$
69. $\lim_{x \to (\pi/2)^-} \frac{\sec x}{\tan x}$
70. $\lim_{x \to 0^+} \frac{\cot x}{\csc x}$
71. $\lim_{x \to \infty} \frac{2^x - 3^x}{3^x + 4^x}$
72. $\lim_{x \to -\infty} \frac{2^x + 4^x}{5^x - 2^x}$
73. $\lim_{x \to \infty} \frac{e^{x^2}}{x e^x}$
74. $\lim_{x \to 0^+} \frac{x}{e^{-1/x}}$

75. Which one is correct, and which one is wrong? Give reasons for your answers.

    a. $\lim_{x \to 3} \frac{x - 3}{x^2 - 3} = \lim_{x \to 3} \frac{1}{2x} = \frac{1}{6}$ **b.** $\lim_{x \to 3} \frac{x - 3}{x^2 - 3} = \frac{0}{6} = 0$

76. Which one is correct, and which one is wrong? Give reasons for your answers.

    a. $\lim_{x \to 0} \frac{x^2 - 2x}{x^2 - \sin x} = \lim_{x \to 0} \frac{2x - 2}{2x - \cos x}$

    $= \lim_{x \to 0} \frac{2}{2 + \sin x} = \frac{2}{2 + 0} = 1$

    b. $\lim_{x \to 0} \frac{x^2 - 2x}{x^2 - \sin x} = \lim_{x \to 0} \frac{2x - 2}{2x - \cos x} = \frac{-2}{0 - 1} = 2$

77. Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

a. $\lim_{x\to 0^+} x \ln x = 0 \cdot (-\infty) = 0$

b. $\lim_{x\to 0^+} x \ln x = 0 \cdot (-\infty) = -\infty$

c. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)} = \dfrac{-\infty}{\infty} = -1$

d. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)}$

$$= \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+} (-x) = 0$$

78. Find all values of c that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

a. $f(x) = x, \quad g(x) = x^2, \quad (a, b) = (-2, 0)$

b. $f(x) = x, \quad g(x) = x^2, \quad (a, b)$ arbitrary

c. $f(x) = x^3/3 - 4x, \quad g(x) = x^2, \quad (a, b) = (0, 3)$

79. Continuous extension Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

80. For what values of a and b is

$$\lim_{x\to 0}\left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x}\right) = 0?$$

T **81. $\infty - \infty$ Form**

a. Estimate the value of

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 + x}\right)$$

by graphing $f(x) = x - \sqrt{x^2 + x}$ over a suitably large interval of x-values.

b. Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $\left(x + \sqrt{x^2 + x}\right)/\left(x + \sqrt{x^2 + x}\right)$ and simplify the new numerator.

82. Find $\lim_{x\to\infty}\left(\sqrt{x^2 + 1} - \sqrt{x}\right)$.

T **83. 0/0 Form** Estimate the value of

$$\lim_{x\to 1}\frac{2x^2 - (3x + 1)\sqrt{x} + 2}{x - 1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

84. This exercise explores the difference between the limit

$$\lim_{x\to\infty}\left(1 + \frac{1}{x^2}\right)^x$$

and the limit

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

a. Use l'Hôpital's Rule to show that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

T **b.** Graph

$$f(x) = \left(1 + \frac{1}{x^2}\right)^x \quad \text{and} \quad g(x) = \left(1 + \frac{1}{x}\right)^x$$

together for $x \geq 0$. How does the behavior of f compare with that of g? Estimate the value of $\lim_{x\to\infty} f(x)$.

c. Confirm your estimate of $\lim_{x\to\infty} f(x)$ by calculating it with l'Hôpital's Rule.

85. Show that

$$\lim_{k\to\infty}\left(1 + \frac{r}{k}\right)^k = e^r.$$

86. Given that $x > 0$, find the maximum value, if any, of

a. $x^{1/x}$

b. x^{1/x^2}

c. x^{1/x^n} (n a positive integer)

d. Show that $\lim_{x\to\infty} x^{1/x^n} = 1$ for every positive integer n.

87. Use limits to find horizontal asymptotes for each function.

a. $y = x \tan\left(\dfrac{1}{x}\right)$ **b.** $y = \dfrac{3x + e^{2x}}{2x + e^{3x}}$

88. Find $f'(0)$ for $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

T **89. The continuous extension of $(\sin x)^x$ to $[0, \pi]$**

a. Graph $f(x) = (\sin x)^x$ on the interval $0 \leq x \leq \pi$. What value would you assign to f to make it continuous at $x = 0$?

b. Verify your conclusion in part (a) by finding $\lim_{x\to 0^+} f(x)$ with l'Hôpital's Rule.

c. Returning to the graph, estimate the maximum value of f on $[0, \pi]$. About where is max f taken on?

d. Sharpen your estimate in part (c) by graphing f' in the same window to see where its graph crosses the x-axis. To simplify your work, you might want to delete the exponential factor from the expression for f' and graph just the factor that has a zero.

T **90. The function $(\sin x)^{\tan x}$** (*Continuation of Exercise 89.*)

a. Graph $f(x) = (\sin x)^{\tan x}$ on the interval $-7 \leq x \leq 7$. How do you account for the gaps in the graph? How wide are the gaps?

b. Now graph f on the interval $0 \leq x \leq \pi$. The function is not defined at $x = \pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for f at $x = \pi/2$? (*Hint:* Use l'Hôpital's Rule to find lim f as $x \to (\pi/2)^-$ and $x \to (\pi/2)^+$.)

c. Continuing with the graphs in part (b), find max f and min f as accurately as you can and estimate the values of x at which they are taken on.