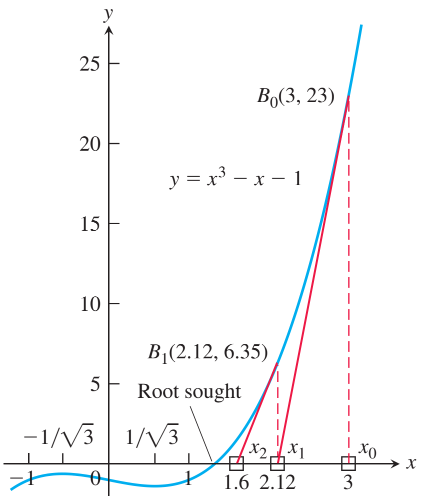


FIGURE 4.52 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root in Example 2.

we use Equation (1) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we obtain the nine-place solution $x_7 = x_6 = 1.3247\ 17957$ in seven steps.

Convergence of the Approximations

In Chapter 10 we define precisely the idea of *convergence* for the approximations x_n in Newton's method. Intuitively, we mean that as the number n of approximations increases without bound, the values x_n get arbitrarily close to the desired root r. (This notion is similar to the idea of the limit of a function $g(t)$ as t approaches infinity, as defined in Section 2.6.)

In practice, Newton's method usually gives convergence with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for x_0. You can test that you are getting closer to a zero of the function by checking that $|f(x_n)|$ is approaching zero, and you can check that the approximations are converging by evaluating $|x_n - x_{n+1}|$.

Newton's method does not always converge. For instance, if

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \geq r, \end{cases}$$

the graph will be like the one in Figure 4.53. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

If Newton's method does converge, it converges to a root. Be careful, however. There are situations in which the method appears to converge but no root is there. Fortunately, such situations are rare.

When Newton's method converges to a root, it may not be the root you have in mind. Figure 4.54 shows two ways this can happen.

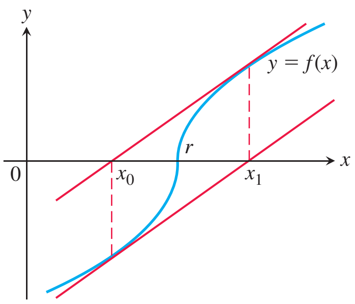


FIGURE 4.53 Newton's method fails to converge. You go from x_0 to x_1 and back to x_0, never getting any closer to r.

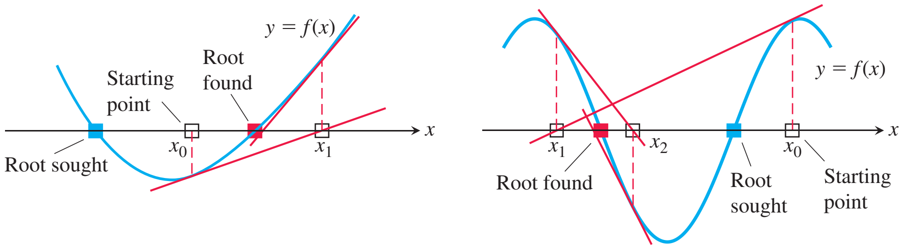


FIGURE 4.54 If you start too far away, Newton's method may miss the root you want.

EXERCISES 4.7

Root Finding

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find x_2.
2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.
3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2.
4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2.
5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.
6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.
7. Use Newton's method to find an approximate solution of $3 - x = \ln x$. Start with $x_0 = 2$ and find x_2.
8. Use Newton's method to find an approximate solution of $x - 1 = \tan^{-1} x$. Start with $x_0 = 1$ and find x_2.
9. Use Newton's method to find an approximate solution of $xe^x = 1$. Start with $x_0 = 0$ and find x_2.

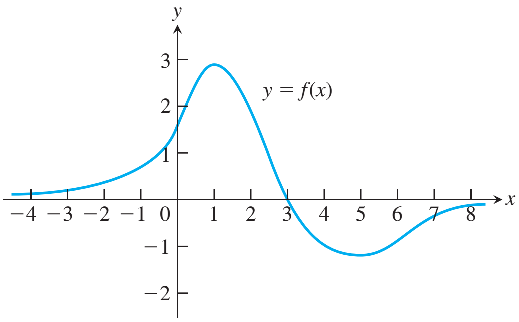
Dependence on Initial Point

10. Using the function shown in the figure, and for each initial estimate x_0, determine graphically what happens to the sequence of Newton's method approximations

a. $x_0 = 0$ **b.** $x_0 = 1$

c. $x_0 = 2$ **d.** $x_0 = 4$

e. $x_0 = 5.5$



11. Guessing a root Suppose that your first guess is lucky, in the sense that x_0 is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to x_1 and later approximations?

12. Estimating pi You plan to estimate $\pi/2$ to five decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

Theory and Examples

13. Oscillation Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

14. Approximations that get worse and worse Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate x_1, x_2, x_3, and x_4. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

15. Explain why the following four statements ask for the same information:

i) Find the roots of $f(x) = x^3 - 3x - 1$.

ii) Find the x-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.

iii) Find the x-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.

iv) Find the values of x where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

16. Locating a planet To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find x_1. (The value of the root is 1.49870 to five decimal places.) Remember to use radians.

T **17. Intersecting curves** The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

T **18. Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

T **19. a.** How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?

b. Use Newton's method to find them.

20. Intersection of curves

a. Does $\cos 3x$ ever equal x? Give reasons for your answer.

b. Use Newton's method to find where.

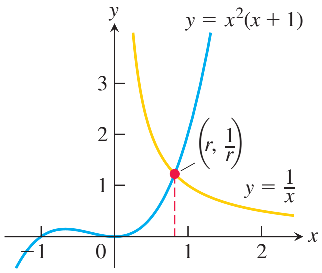
21. Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

T **22. Estimating pi** Estimate π to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

23. Intersection of curves At what value(s) of x does $\cos x = 2x$?

24. Intersection of curves At what value(s) of x does $\cos x = -x$?

25. The graphs of $y = x^2(x + 1)$ and $y = 1/x$ $(x > 0)$ intersect at one point $x = r$. Use Newton's method to estimate the value of r to four decimal places.



26. The graphs of $y = \sqrt{x}$ and $y = 3 - x^2$ intersect at one point $x = r$. Use Newton's method to estimate the value of r to four decimal places.

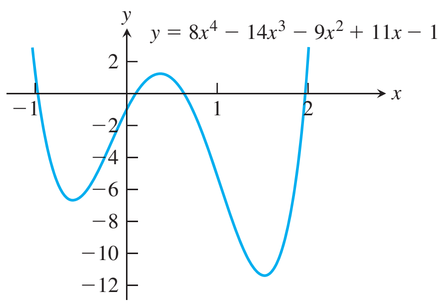
27. Intersection of curves At what value(s) of x does $e^{-x^2} = x^2 - x + 1$?

28. Intersection of curves At what value(s) of x does $\ln(1 - x^2) = x - 1$?

29. Use the Intermediate Value Theorem from Section 2.5 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to five decimal places.

30. Factoring a quartic Find the approximate values of r_1 through r_4 in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$



T **31. Converging to different zeros** Use Newton's method to find the zeros of $f(x) = 4x^4 - 4x^2$ using the given starting values.

a. $x_0 = -2$ and $x_0 = -0.8$, lying in $\left(-\infty, -\sqrt{2}/2\right)$

b. $x_0 = -0.5$ and $x_0 = 0.25$, lying in $\left(-\sqrt{21}/7, \sqrt{21}/7\right)$

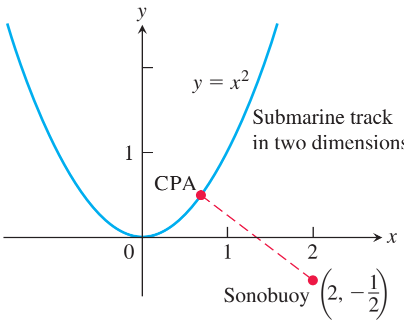
c. $x_0 = 0.8$ and $x_0 = 2$, lying in $\left(\sqrt{2}/2, \infty\right)$

d. $x_0 = -\sqrt{21}/7$ and $x_0 = \sqrt{21}/7$

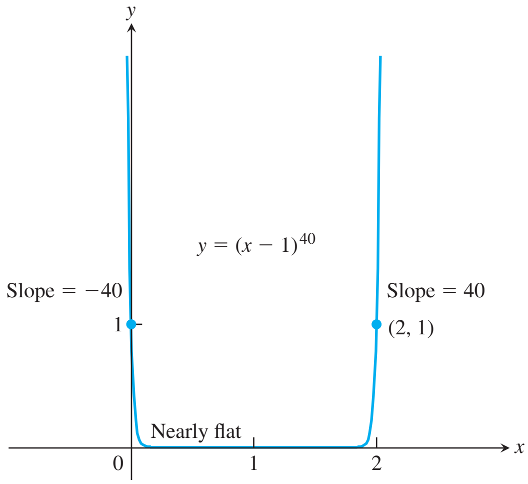
32. The sonobuoy problem In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.

a. Show that the value of x that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.

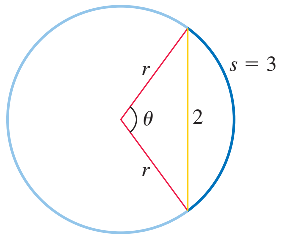
b. Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.



T **33. Curves that are nearly flat at the root** Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$. See the accompanying graph.



34. The accompanying figure shows a circle of radius r with a chord of length 2 and an arc s of length 3. Use Newton's method to solve for r and θ (radians) to four decimal places. Assume $0 < \theta < \pi$.



4.8 Antiderivatives

Many problems require that we recover a function from its derivative, or from its rate of change. For instance, the laws of physics tell us the acceleration of an object falling from an initial height, and we can use this to compute its velocity and its height at any time. More generally, starting with a function f, we want to find a function F whose derivative is f. If such a function F exists, it is called an *antiderivative* of f. Antiderivatives are the link connecting the two major elements of calculus: derivatives and definite integrals.

Finding Antiderivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE 1 Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ **(b)** $g(x) = \cos x$ **(c)** $h(x) = \dfrac{1}{x} + 2e^{2x}$