Math 21A, Winter 2024.

Discussion Problems 6 (Tue., Feb. 27): Solutions

Problem 1. Find all critical points of $y = f(x)$ given by

$$f(x) = x + 5\arctan\frac{1}{x}$$

on the interval $(0,\infty)$. Then find the global maximum and minimum of $y = f(x)$ on the interval $[1,5]$.

Solution.

$$f'(x) = \frac{d}{dx}\left(x + 5\arctan\frac{1}{x}\right) == 1+5\cdot\frac{1}{1+(1/x)^2}\cdot(-1/x^2) = 1 - \frac{5}{1+x^2} = \frac{x^2-4}{1+x^2} = \frac{(x-2)(x+2)}{1+x^2}.$$

The only critical point on $(0,\infty)$ is at $x = 2$. Therefore, we need to evaluate:

- $f(1) = 1 + 5\arctan 1 = 1 + \frac{5\pi}{4} \approx 4.93$
- $f(2) = 2 + 5\arctan\frac{1}{2} \approx 4.32$
- $f(5) = 5 + 5\arctan\frac{1}{5} \approx 5.99$

The global maximum on $[1,5]$ is achived at $x = 5$ with maximal value $f(5) \approx 5.99$ and the global minimum at $x = 2$ with minimal value $f(2) \approx 4.32$.

Problem 2. Find domain and range of the function f given by $f(x) = \sqrt{x^2 - x^4}$.

Solution. Let $g(x) = x^2 - x^4 = x^2(1-x^2)$. Then $g(x) \geq 0$ when $x^2 \leq 1$, that is when x is in $[-1,1]$. The domain of f is therefore $[-1,1]$. To find the range of g, we compute its global maximum and minimum on $[-1,1]$. To do that,

$$g'(x) = 2x - 4x^3 = 2x(1-2x^2),$$

and therefore the three critical points of g on $[-1,1]$ are $x = 0$, $x = 1/\sqrt{2}$, and $x = -1/\sqrt{2}$.

- $g(-1) = 0$
- $g(-1/\sqrt{2}) = 1/2 - 1/4 = 1/4$
- $g(0) = 0$
- $g(1/\sqrt{2}) = 1/2 - 1/4 = 1/4$
- $g(1) = 0$

We conclude that the global maximal value is $1/4$, achieved at both $x = 1/\sqrt{2}$ and $x = -1/\sqrt{2}$. The global minimum value is 0, achieved at three points $x = 0$, $x = 1$ and $x = -1$. As g is continuous, by Intermediate value Theorem, the range of $y = g(x)$ is $[0, 1/4]$. The range of $y = f(x) = \sqrt{g(x)}$, which is also continuous on its domain, consists of square roots of numbers in $[0, 1/4]$, therefore the range of $y = f(x)$ is $[0, 1/2]$.

Answer: the domain of $y = f(x)$ is $[-1, 1]$ and the range of $y = f(x)$ is $[0, 1/2]$.

Problem 3. A particle is moving on a coordinate line. Its position at time $t \geq 0$ is given by $s = f(t)$, where $f(t) = 4t - \cos(2t)$. How many times does the particle visit the origin (i.e., its position is $s = 0$)?

Solution. We have

$$\frac{ds}{dt} = f'(t) = 4 + 2\sin(2t) \geq 2.$$

Thus $f'(t)$ is never zero, and by Rolle's Theorem $f(t)$ can be zero *at most once*. (Because, if it $f(t_1) = f(t_2) = 0$ for different t_1 and t_2, Rolle's Theorem would imply that $f'(t) = 0$ for some t between t_1 and t_2.)

Moreover, f is continuous, $f(0) = -1 < 0$ and $f(\pi/4) = \pi > 0$. By Intermediate Value Theorem, $f(t) = 0$ for some t in $(0, \pi/4)$.

Conclusion: $f(t) = 0$ for *exactly one* t, and consequently the particle visits the origin *exactly once*.

Problem 4. Let $f(x) = x^2(x-4)^{2/3}$. Find the global maximum and the global minimum of $y = f(x)$ on $[0, 5]$ and on $[-4, 4]$.

Solution. Note that the domain of f consists of all real numbers x. We will compute all critical points of f on $(-\infty, \infty)$, and then consider each of the two intervals separately. We have

$$\begin{aligned} f'(x) &= 2x(x-4)^{2/3} + x^2 \cdot \frac{2}{3} \cdot (x-4)^{-1/3} \\ &= \frac{2}{3}(x-4)^{-1/3}(3(x-4) + x) \\ &= \frac{2}{3}(x-4)^{-1/3}(4x - 12) \\ &= \frac{8}{3}(x-4)^{-1/3}(x-3). \end{aligned}$$

(Factor out the lowest power, in this case $(x-4)^{-1/3}$, when you are simplifying expressions like this.)

The critical points are $x = 3$ (when the derivative is zero) and $x = 4$ (when the derivative is undefined). We need to evaluate f at those points, and at the endpoints of the two intervals.

- $f(-4) = -64$
- $f(0) = 0$
- $f(3) = 9$
- $f(4) = 0$
- $f(5) = 25$

On $[-4, 4]$, the maximal value of $f(2.4) \approx 9.85$ is achieved at $x = 2.4$ and the minimal value $f(-4) = -48$ is achieved at $x = -4$.

On On $[0, 5]$, the maximal value of $f(5) = 15$ is achieved at $x = 5$ and the minimal value of $f(0) = f(4) = 0$ is achieved at $x = 0$ and $x = 4$.

Problem 5. Assume that a function f is defined, continuous and differentiable for all x. Give a *precise* argument for your answer on each of the following questions.

(a) If $f(1) = 1$ and $f(2) = 3$, is it possible that $f'(x) > 3$ for all x?

Solution. No. By the mean value theorem, there exists a c in $(1, 2)$, so that

$$f'(c) = \frac{f(2) - f(1)}{2 - 1} = 2.$$

(b) If $f(0) = 5$ and $f'(0) = -1$, is it possible that $f(x) \leq 5$ for all x?

Solution. No. If $f(0) = 5$ and $f(x) \leq 5$ for all x, then f has a global maximum at 0, therefore also a local maximum at 0, and so $f'(0) = 0$.

Problem 6. Consider the function $f(x) = x^4 - 2x^2$ on the domain $D = [0, 2]$.

(a) Find the range of $y = f(x)$ on D.

Solution. As

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

we have only one critical number $x = 1$ in $(0, 2)$. As $f(0) = 0$, $f(1) = -1$ and $f(2) = 8$, the global maximum of f on D is 8 and the global minimum is -1. The range is $[-1, 8]$.

(b) Find the range of $y = f(x)^2$ on D.

Solution. This is the same as finding the range of $y = x^2$ on $[-1, 8]$, so the answer is $[0, 64]$.

(c) Find the range of $y = \cos\left(\frac{\pi}{4} f(x)\right)$ on D.

Solution. This is the same as finding the range of $y = \cos x$ on $[-\frac{\pi}{4}, 2\pi]$, which includes the full interval $[0, 2\pi]$, so the answer is $[-1, 1]$.

(d) Find the range of $y = \sin\left(\frac{\pi}{6} f(x)\right)$ on D.

Solution. This is the same as finding the range of $y = \sin x$ on $[-\frac{\pi}{6}, \frac{8\pi}{6}] = [-\frac{\pi}{6}, \pi + \frac{\pi}{3}]$, so — and this is best seen by drawing the graph of sin on this interval — the answer is $[-\frac{\sqrt{3}}{2}, 1]$.

Note. Some of these solutions required a calculator. A calculator will not be necessary, or allowed, on the exams.