Problem 1 (a)

Graph the following functions using the first and the second derivative. Mark all important points on each graph.

(a) $y = (\ln x)^2 = f(x)$ Domain: $(0, \infty)$

$f'(x) = 2 \ln x \cdot \frac{1}{x}$; $f'(x) = 0$ when $x = 1$, the only critical no.

	$(0, 1)$	$(1, \infty)$
sign of f'	$-$	$+$
f	$\searrow$	$\nearrow$

$(1, 0)$ local and global min

$f''(x) = 2 \cdot \left(\frac{1}{x^2} + \ln x \cdot \left(-\frac{1}{x^2} \right) \right)$

$= \frac{2}{x^2} (1 - \ln x)$

$f''(x) = 0$ when $\ln x = 1$,

that is, $x = e$.

	$(0, e)$	(e, ∞)
sign of f''	$+$	$-$
f	$\smile$ c. up	$\frown$ c. down

$(e, 1)$ inflect. pt.

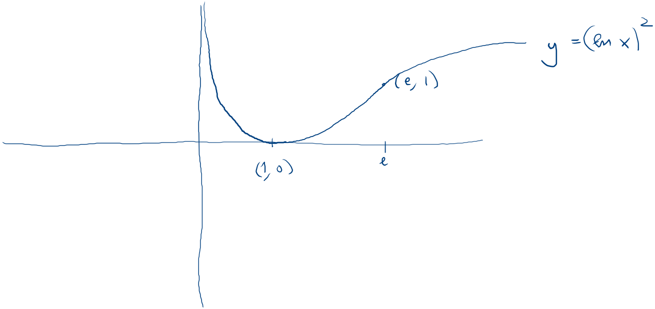
Problem 1 (a)

Graph the following functions using the first and the second derivative. Mark all important points on each graph.

(a) $y = (\ln x)^2$

x-intercept $(1, 0)$

$\lim_{x \to 0^+} (\ln x)^2 = \infty$ $\qquad$ $\lim_{x \to \infty} (\ln x)^2 = \infty$



Problem 1 (b)

(b) $y = \dfrac{e^x}{e^x + 1}$ $= f(x)$ Domain all x

$$f'(x) = \frac{e^x(e^x+1) - e^x \cdot e^x}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2} > 0$$

always increasing $\nearrow$

$$f''(x) = \frac{e^x(e^x+1)^2 - e^x \cdot 2(e^x+1) \cdot e^x}{(e^x+1)^4}$$

$$= \frac{e^x(e^x+1 - 2e^x)}{(e^x+1)^3}$$

$$= \frac{e^x(1 - e^x)}{(e^x+1)^3}$$

$f''(x) = 0$ when $e^x = 1$, that is, $x = 0$.

	$(-\infty, 0)$	$(0, \infty)$
sign of f''	$+$	$-$
f	c. up $\smile$	c. down $\frown$

$\left(0, \frac{1}{2}\right)$

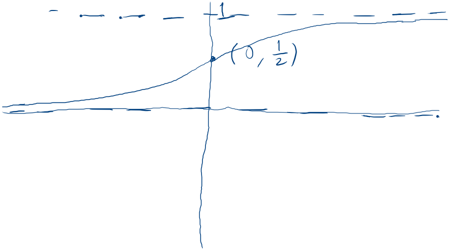
(b) $y = \dfrac{e^x}{e^x + 1}$

$$\lim_{x \to \infty} \frac{e^x \,/ e^x}{e^x + 1 \,/ e^x} = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{e^x}} = 1$$

$y = 1$ horiz. asympt. as $x \to \infty$

$$\lim_{x \to -\infty} \frac{e^x}{e^x + 1} = \frac{0}{0 + 1} = 0$$

$y = 0$ horiz. asympt. as $x \to -\infty$



$y = \dfrac{e^x}{e^x + 1}$

Problem 1(c)

(a) $y = \dfrac{x^2-4}{x+1} = f(x)$ Domain: $x \neq -1$ Intercepts: $(0,-4), (-2,0), (2,0)$.

No problem using quotient rule. An algebraic trick shortens computation.

$$\begin{array}{l} x^2 - 4 : x + 1 = x - 1 \\ -(x^2 + x) \\ \hline \quad -x - 4 \\ \quad -(-x - 1) \\ \hline \qquad -3 \end{array}$$

$$f(x) = x - 1 - \frac{3}{x+1}$$

$$f'(x) = 1 + \frac{3}{(x+1)^2} = 1 + 3(x+1)^{-2}$$

$f'(x)$ DNE at $x = -1$

	$(-\infty, -1)$	$(-1, \infty)$
sign of f'	+	+
f	↗	↗

$$f''(x) = 3(-2)(x+1)^{-3}$$
$$= \frac{-6}{(x+1)^3}$$

$f''(x)$ DNE $x = -1$

$f''(x)$ is never zero

	$(-\infty, -1)$	$(-1, \infty)$
sign of f''	+	−
f	∪ c. up	∩ c. down

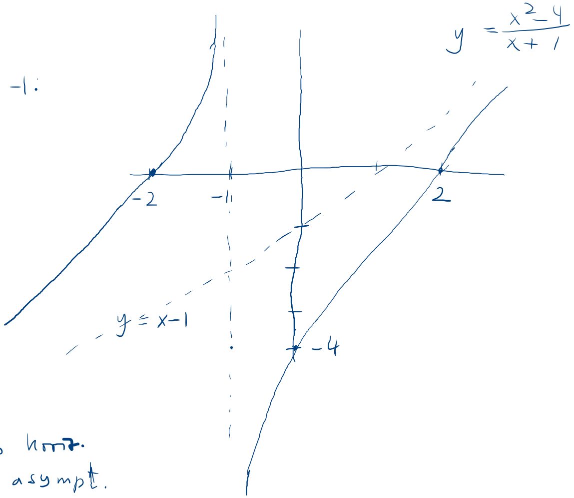
Problem 1(c)

(a) $y = \dfrac{x^2 - 4}{x + 1} = f(x)$

vert. asympt. at $x = -1$:

$\lim_{x \to -1^+} f(x) = -\infty \quad \dfrac{(\approx -3)}{(\text{small} > 0)}$

$\lim_{x \to -1^-} f(x) = \infty \quad \dfrac{(\approx -3)}{(\text{small} < 0)}$



$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

No horiz. asympt.

Problem 2

2. Assume that f has continuous second derivative and that $f''(x) > 0$ for every x.

Can f have a local maximum?

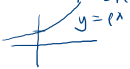
(i.e. $f'(c) = 0$)

2nd derivative test: If $f''(c) > 0$ at a critical pt. c, then the function has a local min there.

No.

Can f have no local minima? Yes, it can.

$f(x) = e^x$ $\quad f''(x) = e^x > 0$. But f is always increasing ($f'(x) = e^x > 0$), so it has no local extrema



Can f have more than one local minimum?

As $f''(x)$ is never zero, $f'(x)$ can have at most one zero, so f can have at most one critical pt., and so at most one local extremum.

Problem 2

Solution summary.

Can f have a local maximum?
No: any critical point must be a local minimum, by the second derivative test.

Can f have no local minima?
Yes: for example $f(x) = e^x$ has $f''(x) = e^x > 0$ but is always increasing.

Can f have more than one local minimum?
No: by Rolle, f' has *at most one x-intercept* (as its derivative f'' is always positive), and therefore f has at most one critical point, so it can have at most one local extremum of any kind.