Math 21A, Winter 2024.

Solutions To Discussion Problems 8 (Tue., Mar. 12)

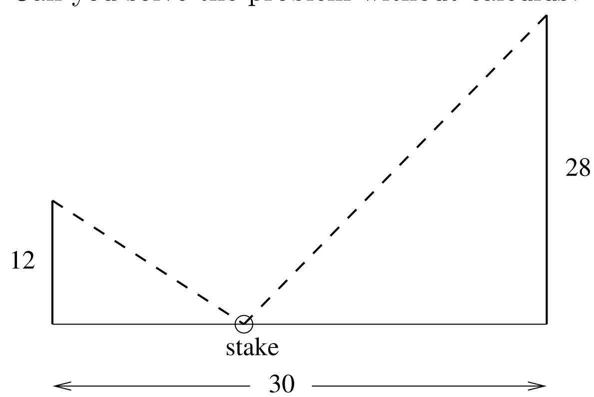
These problems are good preparation for the final exam.

1. Find the point on the graph of $y = \sqrt{x}$ which is nearest to the point $(4, 0)$.
Solution. Let $f(x)$ be the square of the distance between the point $(x, \sqrt{x})$ and $(4, 0)$, that is,

$$f(x) = (x-4)^2 + (\sqrt{x} - 0)^2 = x^2 - 8x + 16 + x = x^2 - 7x + 16,$$

and we need to find the global minimum of this function for $x \geq 0$ (which is the domain of $\sqrt{x}$). Then $f'(x) = 0$, $2x - 7 = 0$, has the only solution $x = 7/2$. As f is a quadratic parabola with positive leading coefficient, it has a global minimum at $x = 7/2$. The closest point is $(7/2, \sqrt{7/2})$.

2. Two posts, one 12 feet hight and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake on the ground and running to the top of each post.
(a) Where should the stake be placed so that the least amount of wire is used?
(b) Can you solve the problem without calculus? (*Hint*: reflect one of the posts.)



Solution. Let x be the distance from the shorter post to the stake. The function to minimize is

$$f(x) = \sqrt{12^2 + x^2} + \sqrt{(30-x)^2 + 28^2}$$

on $x \in [0, 30]$. We have

$$f'(x) = \frac{1}{2}(12^2 + x^2)^{-1/2} \cdot 2x + \frac{1}{2}((30-x)^2 + 28^2)^{-1/2} \cdot 2(30-x)(-1)$$

We rewrite the equation $f'(x) = 0$ into

$$(12^2 + x^2)^{-1/2} \cdot x = ((30-x)^2 + 28^2)^{-1/2} \cdot (30-x),$$

square

$$(12^2 + x^2)^{-1} \cdot x^2 = ((30-x)^2 + 28^2)^{-1} \cdot (30-x)^2,$$

multiply by $(12^2+x^2)\cdot((30-x)^2+28^2)$

$$((30-x)^2+28^2)\cdot x^2=(12^2+x^2)\cdot(30-x)^2,$$

cancel $x^2\cdot(30-x)^2$

$$28^2\cdot x^2=12^2\cdot(30-x)^2,$$

take the square root

$$28\cdot x=12\cdot(30-x),$$

rewrite

$$40\cdot x=12\cdot 30,$$

and finally solve to get

$$x=9.$$

To check that $x=9$ gives the global minimum, we verify

- $f(0)=12+\sqrt{30^2+28^2}\approx 53.04$;
- $f(9)=\sqrt{12^2+9^2}+\sqrt{21^2+28^2}=50$;
- $f(30)=\sqrt{12^2+30^2}+28\approx 60.31$.

So we see that $f(9)$ is the smallest of these.

An alternative way to check that $x=9$ gives the global minimum is to verify that $f'(x)<0$ on $(0,9)$ (as we can see that $f'(x)<0$ when x is close to 0), and $f'(x)>0$ on $(9,30)$ (as we can see that $f'(x)>0$ when x is close to 30), so f is decreasing on $(0,9)$ and increasing on $(9,30)$.

(b) Can you solve the problem without calculus? (*Hint*: reflect one of the posts.)

Solution. If you reflect, say, the longer post, together with the wire between the stake and the longer post, then the wire of minimal length between the top of the shorter post and the top of the reflected longer post must be a straight line, as it is the shortest curve between the two points. The line that that connects $(0,12)$ and $(30,-28)$ is

$$y-12=-\frac{40}{30}x$$

which crosses the x-axis at the solution x of the equation

$$-12=-\frac{40}{30}x,$$

that is, at $x=9$.

3.(This is Problem 53(a,b,c,e) in Section 4.6 of the book, on distance between two ships.) At noon, ship A was 12 nautical miles due north of ship B. Ship A was sailing south at 12 knots (nautical miles per hour) and continued to do so all day. Ship B was sailing east at 8 knots and continued to do so all day.

(a) Start counting time with $t=0$ at noon and express the distance s between the ships as a function of t.

Solution. Place the coordinate system so that the initial position of ship B is at the origin and ship A is on the y-axis. At time t, $(8t,0)$ is the position of ship B and $(0,12-12t)$ is the position of ship A.

The distance between them is

$$s=\sqrt{(0-8t)^2+(12-12t-0)^2}=\sqrt{64t^2+144(1-t)^2}.$$

(b) How rapidly was the distance between the ships changing at noon? One hour later?

Solution. We can just differentiate the expression in (a) to get

$$\frac{ds}{dt} = \frac{2 \cdot 64t - 2 \cdot 144(1-t)}{2\sqrt{64t^2 + 144(1-t)^2}} = \frac{64t - 144(1-t)}{\sqrt{64t^2 + 144(1-t)^2}} = \frac{208t - 144}{\sqrt{64t^2 + 144(1-t)^2}}$$

and plug in $t = 0$ and $t = 1$ to get -12 and 8, respectively.

(c) The visibility that day was 5 nautical miles. Did the ships ever sight each other?

Solution. Let's compute the global minimum of s for t in $[0, \infty)$. The equation $ds/dt = 0$, has one solution, $t = 144/208 = 9/13 =\approx 0.6923$, which is the only critical point. Moreover, $dr/dt < 0$ on $[0, 9/13)$ and $dr/dt > 0$ on $(9/13, \infty)$. So s has global minimum at $t = 9/13$, at which time s equals about 6.6564, so the answer is no.

(d) Compute $\lim_{t\to\infty} \frac{ds}{dt}$.

Solution. By dividing the top and bottom by t, we get

$$\lim_{t\to\infty} \frac{ds}{dt} = \frac{208}{\sqrt{64 + 144}} = \frac{208}{\sqrt{208}} = \sqrt{208},$$

the square root of the sum of the squares of ships' speeds.

4. Compute the following limits:

(a) $\lim_{x\to 0} \frac{1 - \cos(2x)}{xe^x - x}$

Solution. We have 0/0, so by L'Hopital the limit equals

$$\lim_{x\to 0} \frac{2\sin(2x)}{e^x + xe^x - 1}$$

This is still 0/0, so by L'Hopital again,

$$\lim_{x\to 0} \frac{4\cos(2x)}{e^x + e^x + xe^x} = 2$$

The answer is 2.

(b) $\lim_{x\to 0} (1 + 2x)^{5/x}$

Solution. Take ln. Then, we need to compute

$$\lim_{x\to 0} \frac{5\ln(1 + 2x)}{x}$$

We have 0/0, so by L'Hopital this limit equals

$$\lim_{x\to 0} \frac{5 \cdot \frac{1}{1+2x} \cdot 2}{1} = 10$$

and the answer is e^{10}.

(c) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{2}{\sin(2x)}\right)$

Solution. Rewrite, by using the common denominator

$$\lim_{x\to 0} \frac{\sin(2x) - 2x}{x\sin(2x)}$$

We can start doing L'Hopital at this point, but it is quicker to write

$$\lim_{x\to 0} \frac{\sin(2x) - 2x}{2x^2} \cdot \frac{2x}{\sin(2x)}$$

and the second factor goes to 1 so we can drop it, and then need to compute

$$\lim_{x\to 0} \frac{\sin(2x) - 2x}{2x^2}$$

which is 0/0 so by L'Hopital it equals

$$\lim_{x\to 0} \frac{2\cos(2x) - 2}{4x}$$

which is still 0/0 so by L'Hopital it equals

$$\lim_{x\to 0} \frac{-4\sin(2x)}{4} = 0.$$

The answer is 0.

(d) $\lim_{x\to 0+} (e^x - 1)\ln x$

Solution. One possibility would be to write this is

$$\lim_{x\to 0+} \frac{\ln x}{(e^x - 1)^{-1}}$$

and we have ∞/∞ so by L'Hopital this equals

$$\lim_{x\to 0+} \frac{x^{-1}}{-(e^x - 1)^{-2}e^x} = \lim_{x\to 0+} \frac{-(e^x - 1)^2}{xe^x} = \lim_{x\to 0+} \frac{-(e^x - 1)^2}{x}$$

because we can drop the factor e^x from the denominator as it goes to 1. This is 0/0 so by L'Hopital it equals

$$\lim_{x\to 0+} \frac{-2(e^x - 1)e^x}{1} = 0.$$

The answer is 0.

Another, quicker, possibility is to write this limit as a product like this: this is

$$\lim_{x\to 0+} \frac{\ln x}{1/x} \cdot \lim_{x\to 0+} \frac{e^x - 1}{x}$$

which holds provided both of these limits exist and are finite. But the first expression is $-\infty/\infty$ and the second is 0/0, so we can apply L'Hopital to each separately to get

$$\lim_{x\to 0+} \frac{x^{-1}}{-x^{-2}} \cdot \lim_{x\to 0+} \frac{e^x}{1}$$

and the first limit is 0 (as the expression is $-x$), while the second is 1. We get the same answer 0, of course.

(e) $\lim_{x\to\infty} (3^x + 4^x)^{1/x}$

Solution. After taking ln, we get

$$\lim_{x\to\infty} \frac{\ln(3^x + 4^x)}{x}$$

which is ∞/∞ so by L'Hopital it equals

$$\lim_{x\to\infty} \frac{(3^x + 4^x)^{-1}(3^x \ln 3 + 4^x \ln 4)}{1} = \lim_{x\to\infty} \frac{3^x \ln 3 + 4^x \ln 4}{3^x + 4^x}$$

and now divide the top and bottom by the largest quantity 4^x to get

$$\lim_{x\to\infty} \frac{(3/4)^x \ln 3 + \ln 4}{(3/4)^x + 1}$$

which goes to $\ln 4$. The answer is $e^{\ln 4} = 4$.

Another way, which is a quicker one, is to use the sandwich theorem:

$$4 = (4^x)^{1/x} \le (3^x + 4^x)^{1/x} \le (2 \cdot 4^x)^{1/x} = 2^{1/x} \cdot 4,$$

and $2^{1/x}$ goes to $2^0 = 1$, so our quantity is sandwiched between 4 and an expression that goes to 4, so the limit must be 4.

(f) $\lim_{x\to 0} \frac{e^{-1/x^2}}{x}$

Solution. This is 0/0 but we'd get nowhere by applying L'Hopital in this form, no matter how many times. Instead, we take $z = 1/x$, assume first that $x > 0$ so that $z \to \infty$, and rewrite the limit as

$$\lim_{z\to\infty} \frac{z}{e^{z^2}}$$

which is ∞/∞, so by L'Hopital it equals

$$\lim_{z\to\infty} \frac{1}{2z\, e^{z^2}} = 0$$

Exactly the same computation applies is $x < 0$ and so $z \to -\infty$. Therefore, the answer is 0.