Math 21A, Fall 2022.
Dec. 5. 2022.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions.

Make sure that you have a total of 11 pages (including this one) with 8 problems.

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

1. Please be careful: on each of the parts (a)–(c) below, you will receive little or no credit if you make a differentiation mistake, even a small one.

(a) Compute the derivative of the function $y = \dfrac{\sqrt{x}+x}{x^2+1}$. *Do not simplify!*

$$y' = \frac{(\frac{1}{2}x^{-1/2}+1)(x^2+1) - (\sqrt{x}+x)\cdot 2x}{(x^2+1)^2}$$

(b) Compute the derivative of the function $y = \ln(4 + \arcsin(x^4))$. *Do not simplify!*

$$y' = \frac{1}{4+\arcsin(x^4)} \cdot \frac{1}{\sqrt{1-x^8}} \cdot 4x^3$$

(c) Find the equation of the tangent line to the curve $(2x+y)^5 + xy^2 - y = 0$ at the point $(0,1)$. Give the answer in the slope-intercept form.

$$5(2x+y)^4(2+y') + y^2 + 2xyy' - y' = 0$$

$x=0$, $y=1$

$$5(2+y') + 1 - y' = 0$$

$$11 + 4y' = 0 \qquad y' = -\frac{11}{4} \leftarrow \text{slope}$$

$$y - 1 = -\frac{11}{4}x$$

$$\underline{\underline{y = -\frac{11}{4}x + 1}}$$

2. Compute the following limits, *in any correct way* you can. Give each answer as a finite number, $+\infty$ or $-\infty$.

(a) $\lim\limits_{x\to\infty} \dfrac{x+\sin(x)}{\sqrt{9x^2+1}}$ $\begin{matrix} /x \\ /x \end{matrix}$

$$= \lim_{x\to\infty} \frac{1+\underbrace{\frac{\sin x}{x}}_{\to 0}}{\sqrt{9+\frac{1}{x^2}}} = \frac{1}{3}$$

(b) $\lim\limits_{x\to0^+} \dfrac{e^x-x-5}{e^x-1}$ $= -\infty$

$$\frac{(\approx -4)}{(\text{small} > 0)}$$

(c) $\lim\limits_{x\to0} \dfrac{(e^x-1)^2}{1-\cos(3x)}$ $\overset{(L'H.)}{=} \lim\limits_{x\to0} \dfrac{2(e^x-1)\overbrace{e^x}^{\to 1}}{3\sin(3x)}$

$$\left(\frac{0}{0}\right)$$

$$= \frac{2}{3}\lim_{x\to0} \frac{(e^x-1)}{\sin(3x)} = \frac{2}{3}\lim_{x\to0} \frac{\overbrace{e^x}^{\to 1}}{3\underbrace{\cos(3x)}_{\to 1}} = \frac{2}{9}$$

$$\left(\frac{0}{0}\right)$$

$$f'(x) = \begin{cases} b & x<a \\ -\frac{1}{x^2} & x>a \end{cases}$$

3. Consider the function

$$f(x) = \begin{cases} bx+1 & x<a \\ \frac{1}{x} & x \geq a \end{cases}$$

(a) Determine the numbers a and b so that $y = f(x)$ is differentiable for all x.

Cont. at a: $ab + 1 = \frac{1}{a}$

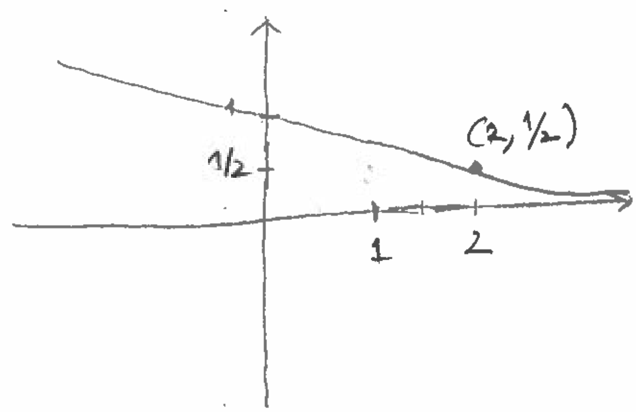
diff. at a: $b = -\frac{1}{a^2}$

$a \cdot \left(-\frac{1}{a^2}\right) + 1 = \frac{1}{a}$

$\frac{2}{a} = 1$, $\underline{a=2}$ $\qquad \underline{b = -\frac{1}{4}}$

$$f(x) = \begin{cases} -\frac{1}{4}x+1 & x<2 \\ \frac{1}{x} & x \geq 2 \end{cases}, \quad f'(x) = \begin{cases} -\frac{1}{4} & x<2 \\ -\frac{1}{x^2} & x \geq 2 \end{cases}$$

(b) Assume the values of a and b obtained in (a). Sketch the graph of the function f using the first derivative.



(c) Assume the values of a and b obtained in (a). Is f one-to-one on $(-\infty, \infty)$?

f is always decreasing ($f'(x) < 0$ for all x)

so it <u>is</u> one-to-one.

4. *In all parts of this problem,* the function f is given by $f(x) = \dfrac{8(x-2)}{x^2} = 8x^{-1} - 16x^{-2}$. *Assume the domain is $x > 0$, that is, the interval $(0, \infty)$.*

(a) Compute $\lim_{x \to 0+} f(x)$ and $\lim_{x \to \infty} f(x)$.

$\lim_{x \to 0+} f(x) = \dfrac{(\approx -16)}{(\text{small} > 0)} = -\infty$

$\lim_{x \to \infty} f(x) = 0$

(b) Determine the intervals on which $y = f(x)$ is increasing and the intervals on which it is decreasing. Identify all local extrema.

$f'(x) = -8x^{-2} + 32x^{-3} = -8x^{-3}(x-4)$ c.p.: $x=4$

	$(0,4)$	$(4,\infty)$
sign of f'	$+$	$-$
f	$\nearrow$	$\searrow$

$(4, 1)$ local max

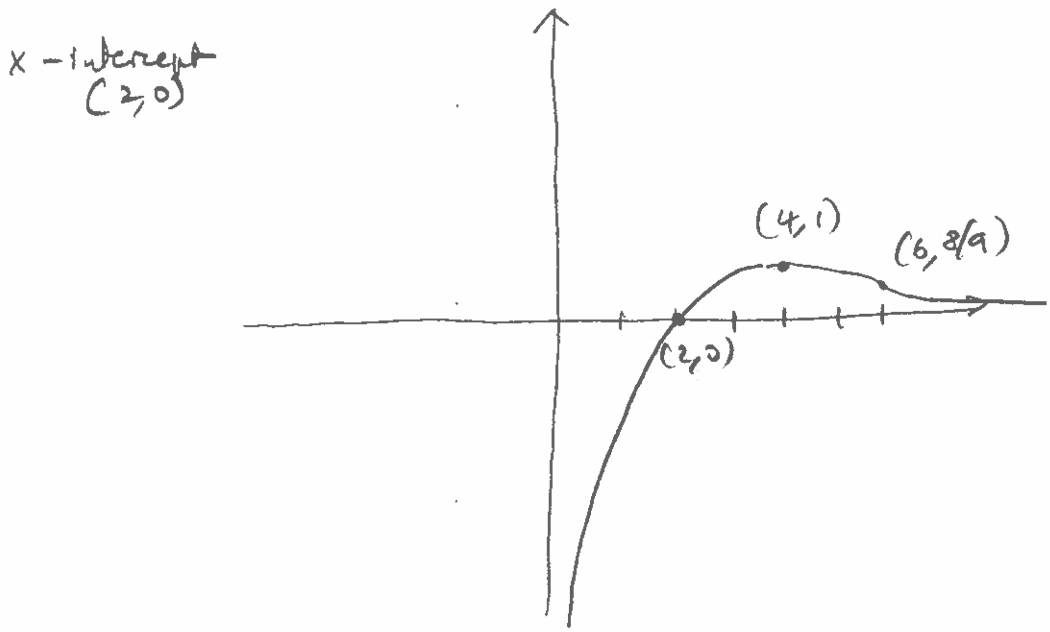
(c) Determine the intervals on which $y = f(x)$ is concave up and the intervals on which it is concave down. Identify all inflection points. (*Note:* $3 \cdot 32 = 6 \cdot 16$.)

$f''(x) = 16x^{-3} - 3 \cdot 32x^{-4} = 16x^{-4}(x-6) = 0$ at $x=6$

	$(0,6)$	$(6,\infty)$
sign of f''	$-$	$+$
f	$\frown$	$\smile$
	c. down	c. up

$(6, \frac{8}{9})$ infl. pt.

(d) (*Still*: the function f is given by $f(x) = \dfrac{8(x-2)}{x^2} = 8x^{-1} - 16x^{-2}$, with domain $(0, \infty)$.) Sketch the graph of $y = f(x)$. Label all points of importance on the graph.



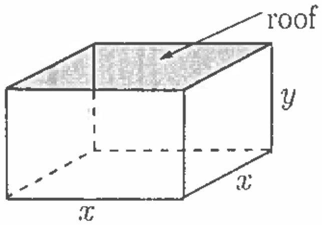
(e) Determine the domain and range of the composite function $y = \sqrt{f(x)}$.

<u>Domain</u>: all x where $f(x) \geq 0$, that is, $x \geq 2$
or $[2, \infty)$.

<u>Range</u>: On $[2, \infty)$, the range of f is $[0, 1]$
and so the range of $y = \sqrt{f(x)}$ is also $[0, 1]$.

5. A construction contractor needs to build a small rectangular storage box with a square base, flat roof and volume 16 cubic feet. Every square foot of side walls costs $1 (that is, the cost of side walls is the area multiplied by $1), and every square foot of the roof costs $4. Assume that the base does not need to be built, so it costs nothing.

(a) Determine the dimensions of the box that will minimize the building costs. *Justify all your conclusions.*



$$C = 4xy + 4x^2$$

$$V = x^2 y = 16 \qquad y = \frac{16}{x^2}$$

$$C = 4\left(\frac{16}{x} + x^2\right) \leftarrow \text{minimize for } x \text{ in } (0,\infty)$$

$$\frac{dC}{dx} = 4\left(-\frac{16}{x^2} + 2x\right) = 8\,\frac{x^3 - 8}{x^2}$$

c.p. $\underline{x = 2}$

	$(0,2)$	$(2,\infty)$
sign of $\frac{dC}{dx}$	$-$	$+$
C	$\searrow$	$\nearrow$

global min at $\underline{\underline{x=2}}$

$\underline{\underline{y=4}}$

(b) Now assume that the height of the box is restricted to be between 1 and 2 feet, that is, in the interval $[1,2]$. With this additional restriction, what dimensions of the box minimize cost?

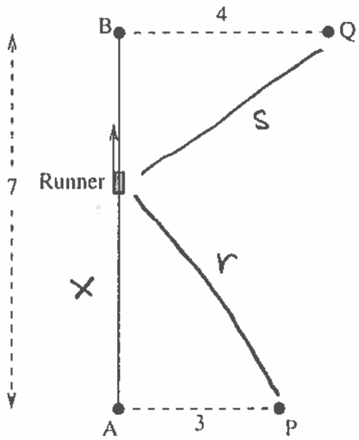
$y = \frac{16}{x^2}$ is in $[1,2]$ and so

x is in $[2\sqrt{2}, 4]$

C <u>increases</u> on this interval, and so the cost is minimized at $\underline{\underline{x = 2\sqrt{2}}}$, $\underline{\underline{y = 2}}$.

6. A runner is running on a straight trail of length 7 miles from point A to point B. Point P is located off the trail; its closest point A on the line AB is 3 miles away. Point Q is also located off the trail; its closest point B on the line AB is 4 miles away. At some time, the distance between the runner and A is 4 miles; at the same time, the distance between the runner and P is increasing at the rate of 8 miles per hour. Find the following rates at this instance.

(a) The velocity of the runner.



$$r^2 = x^2 + 9$$

$$r = \sqrt{x^2+9}$$

$$\frac{dr}{dt} = \frac{1}{\cancel{2}\sqrt{x^2+9}} \cdot \cancel{2}x \frac{dx}{dt}$$

Plug in: $x = 4$, $\frac{dr}{dt} = 8$

$$8 = \frac{1}{5} 4 \frac{dx}{dt}$$

$$\frac{dx}{dt} = 10 \text{ (mph)}$$

(b) The rate of change of the distance between the runner and point Q.

$$s^2 = (7-x)^2 + 16$$

$$s = \sqrt{(7-x)^2+16}$$

$$\frac{ds}{dt} = \frac{1}{\cancel{2}\sqrt{(7-x)^2+16}} \cdot \cancel{2}(7-x) \cdot (-1) \frac{dx}{dt}$$

$$= -\frac{7-x}{\sqrt{(7-x)^2+16}} \frac{dx}{dt}$$

Plug in $x=4$, $\frac{dx}{dt} = 10$: $\frac{ds}{dt} = -\frac{3}{5} \cdot 10$

$$= -6 \text{ (mph)}$$

7. Throughout this problem, the function f is given by $f(x) = \dfrac{1+4x^2}{x^2} = x^{-2}+4$. *Assume the domain of f is $x > 0$ throughout this problem.*

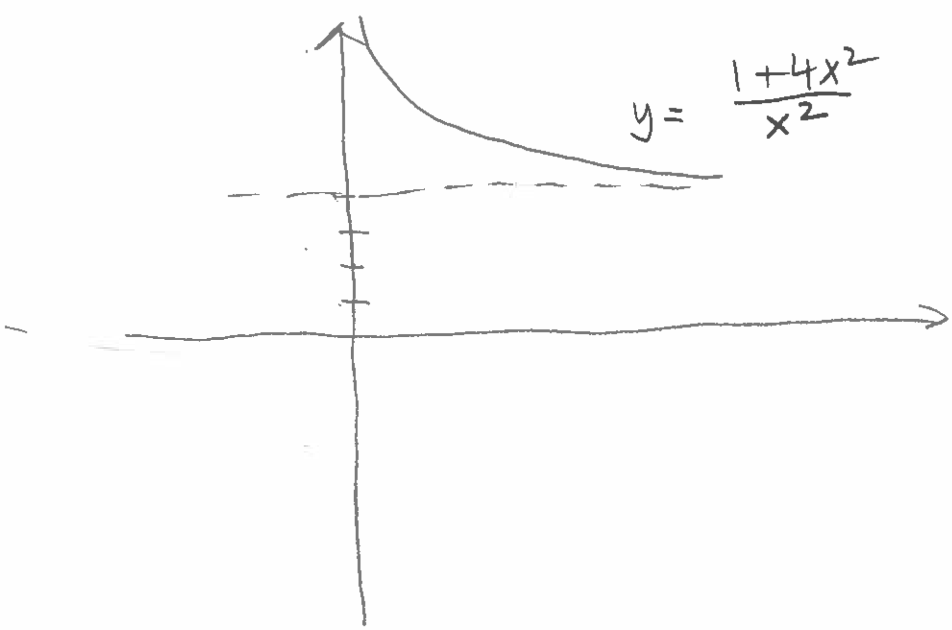
(a) Identify the monotonicity and concavity properties of the function f, compute the relevant limits, and sketch the graph of f.

$f'(x) = -2x^{-3} < 0$ $\quad$ f is always decreasing

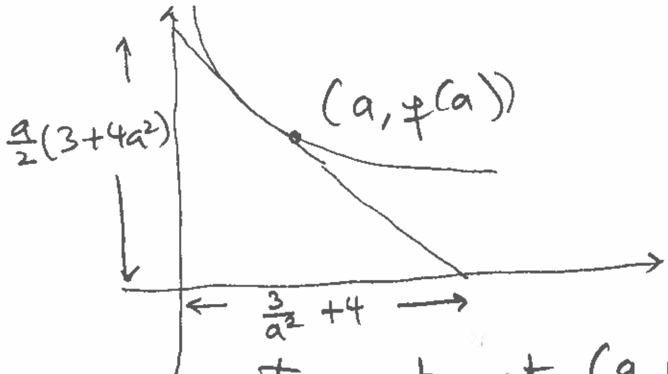
$f''(x) = 6x^{-4} > 0$ $\quad$ f is always concave up

$\lim_{x \to 0+} f(x) = \infty \qquad \dfrac{(\approx 1)}{(\text{small} > 0)}$

$\lim_{x \to \infty} f(x) = 4 \qquad y = 4$ horizontal asymptote



(b) (Still $f(x) = \dfrac{1+4x^2}{x^2} = x^{-2} + 4$ and $x > 0$.) What is the smallest possible area of the triangle formed in the first quadrant by the x-axis, y-axis, and a tangent to the graph of f? (The first quadrant consists of points (x, y) on the coordinate plane with $x \geq 0$ and $y \geq 0$.)



$$f'(x) = -2x^{-3}$$

Tangent at $(a, f(a))$: $y - (a^{-2}+4) = -2a^{-3}(x-a)$

$$y = -\frac{2}{a^3}x + \frac{3}{a^2} + 4$$

$x = 0$: $y = \frac{3}{a^2} + 4 = \frac{3+4a^2}{a^2}$

$y = 0$: $x = \frac{a^3}{2}\left(\frac{3}{a^2} + 4\right) = \frac{a}{2}(3+4a^2)$

Area : $A = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{(3+4a^2)^2}{a} = \frac{1}{4}\frac{(3+4a^2)^2}{a}$

Minimize for $a > 0$.

$$\frac{dA}{da} = \frac{1}{4}\,\frac{2(3+4a^2)\,8a\cdot a - (3+4a^2)^2}{a^2}$$

$$= \frac{1}{4}\,\frac{(3+4a^2)(16a^2 - 3 - 4a^2)}{a^2} = \frac{1}{4}\,\frac{(3+4a^2)(12a^2-3)}{a^2}$$

c.p. $a^2 = \frac{1}{4}$, $a = \frac{1}{2}$

	$(0, \frac{1}{2})$	$(\frac{1}{2}, \infty)$
$\frac{dA}{da}$	$\searrow$	$\nearrow$

$\Rightarrow$ Global min at $a = \frac{1}{2}$,

$$A = \frac{1}{4} \cdot \frac{(3 + 4 \cdot \frac{1}{4})^2}{1/2}$$

$$= \underline{\underline{8}}\,.$$

8. Provide straightforward, and *fully justified*, answers to the following questions. In each of them, assume that $y = f(x)$ is a continuous function defined for all x, and $f'(x)$ and $f''(x)$ exist and are continuous for all x. (*Note*: assumptions in (a) apply only to (a); the same is true for (b) and (c).)

(a) $f(-1) = 0$ and $f'(x) \geq 1$ for all x. Is it possible that $f(2) = 2$?

If $f(2) = 2$, by MVT, there is a c in $(-1, 2)$ so that

$$f'(c) = \frac{f(2) - f(-1)}{2-(-1)} = \frac{2-0}{3} = \frac{2}{3} < 1$$

But $f'(x) \geq 1$ for all x, so it is not possible.

(b) $f(0) = 1$ and $f'(0) = 0$. Is it possible that f has no x-intercept? (Either: prove that it is not possible; or give an example of such a function as an expression in x.)

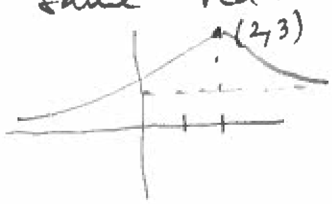
Yes. For example, $f(x) = 1 + x^2$; then $f'(x) = 2x$

$f(0) = 1$,

$f'(0) = 0$;

(c) The only solution to $f'(x) = 0$ is $x = 2$. Moreover, $f(2) = 3$, $\lim_{x\to-\infty} f(x) = 0$, and $\lim_{x\to\infty} f(x) = 1$. Identify the range of f.

The derivative does not change sign on $(-\infty, 2)$ so the function must increase there. For the same reason, the function must decrease on $(2, \infty)$.



Range: $(0, 3]$.