Math 21A, Winter 2024.
Mar. 18, 2024.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions.

Make sure that you have a total of 11 pages (including this one) with 8 problems.

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

1. Please be careful; on each of the parts (a)–(c) below, you will receive little or no credit if you make a differentiation mistake, even a small one.

(a) Compute the derivative of the function $y = \arctan(\sin x + \ln x)$. *Do not simplify!*

$$y' = \frac{1}{1+(\sin x + \ln x)^2} \cdot \left(\cos x + \frac{1}{x}\right)$$

(b) Compute the derivative of the function $y = e^{\sqrt{1+2x}}$. *Do not simplify!*

$$y' = e^{\sqrt{1+2x}} \cdot \frac{1}{2}(1+2x)^{-1/2} \cdot 2$$

(c) Find the equation of the tangent line to the curve $\sqrt{x^2+y^5} + 2x^2\cos y + x = 1$ at the point $(0,1)$. Give the answer in the slope-intercept form.

$$\frac{1}{2}(x^2+y^5)(2x+5y^4y')$$
$$+ 4x\cos y - 2x^2\sin y + 1 = 0$$

Plug in $x=0,\ y=1$:

$$\frac{1}{2}5y' + 1 = 0 \qquad y' = -\frac{2}{5}$$

$$y - 1 = -\frac{2}{5}x$$

$$y = -\frac{2}{5}x + 1$$

2. Compute the following limits, *in any correct way* you can. Give each answer as a finite number, $+\infty$ or $-\infty$.

(a) $\lim_{x\to 3} \frac{x-\sqrt{5x-6}}{x-3}$ $\qquad \frac{(x+\sqrt{5x-6})}{(x+\sqrt{5x-6})}$ $\qquad$ (L'Hopital also works!)

$$= \lim_{x\to 3} \frac{x^2-5x+6}{(x-3)(x+\sqrt{5x-6})}$$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x+\sqrt{5x-6})} = \frac{1}{3+3} = \underline{\underline{\frac{1}{6}}}$$

(b) $\lim_{x\to 0} \frac{\cos(2x)-\cos x}{x^2}$.

$\left(\frac{0}{0}\right)$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-2\sin(2x)+\sin x}{2x}$$

$\left(\frac{0}{0}\right)$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-4\cos(2x)+\cos x}{2}$$

$$= \underline{\underline{-\frac{3}{2}}}$$

(c) $\lim_{x\to 0+} \frac{x^2+e^{x^2}}{\sqrt{x}} = \underline{\underline{\infty}}$

$$\frac{(\approx 1)}{(\text{small} > 0)}$$

$$f'(x) = \begin{cases} 2ax & x < 1 \\ -\frac{2}{x^2} & x > 1 \end{cases}$$

3. Consider the function

$$f(x) = \begin{cases} ax^2 + b, & x < 1 \\ \frac{2}{x}, & x \geq 1 \end{cases}$$

f is cont. and diff. for all $x \neq 1$

(a) Determine the numbers a and b so that $y = f(x)$ is differentiable for all x.

Cont. at $x=1$: $a + b = 2$

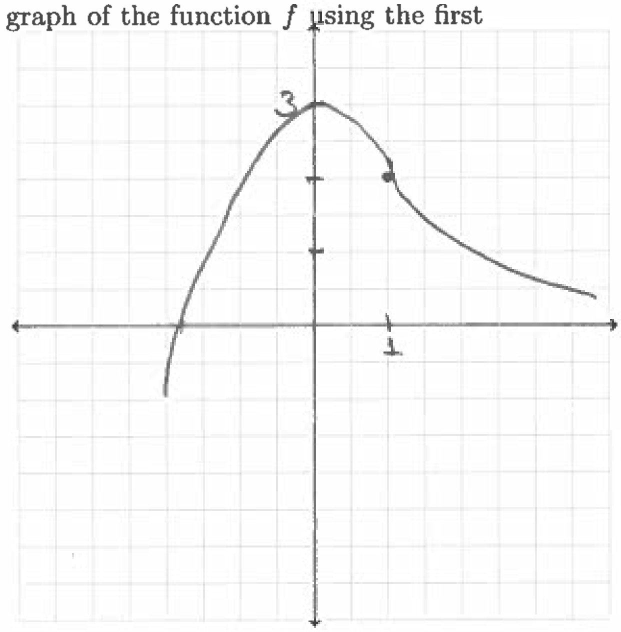
Diff. at $x=1$: $2a = -2$

$$\boxed{\begin{matrix} a = -1 \\ b = 3 \end{matrix}}$$

(b) Assume the values of a and b obtained in (a). Sketch the graph of the function f using the first derivative and determine its range.

$$f(x) = \begin{cases} -x^2 + 3 & x < 1 \\ \frac{2}{x} & x \geq 1 \end{cases}$$

$$f'(x) = \begin{cases} -2x & x < 1 \\ -\frac{2}{x^2} & x \geq 1 \end{cases}$$



f is quadratic parabola for $x < 1$ and is decreasing for $x \geq 0$, with $\lim_{x \to \infty} f(x) = 0$.

Range: $(-\infty, 3]$, i.e. $y \leq 3$.

(c) Determine the domain and range of the composite function $y = f\left(\frac{2}{\pi} \arcsin x\right)$.

The range of $\arcsin x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

so the range of $\frac{2}{\pi} \arcsin x$ is $[-1, 1]$

On $[-1, 1]$, the range of f is $\underline{\underline{[2, 3]}}$.

4. *In all parts of this problem*, the function f is given by $f(x) = \dfrac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$.

(a) Determine the domain of $y = f(x)$. Compute $\lim_{x \to 0+} f(x)$ and $\lim_{x \to \infty} f(x)$.

Domain: $(0, \infty)$

$$\lim_{x \to 0+} f(x) = \infty \qquad \left(\frac{(\approx 1)}{(\text{small} > 0)} \right)$$

$$\lim_{x \to \infty} f(x) = \infty$$

(b) Determine the intervals on which $y = f(x)$ is increasing and the intervals on which it is decreasing. Identify all local extrema.

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2}x^{-3/2}(x-1)$$

c.p. $x = 1$

	$(0, 1)$	$(1, \infty)$
sign of f'	$-$	$+$
f	$\searrow$	$\nearrow$

$(1, 2)$ local (and global) minimum

(c) Determine the intervals on which $y = f(x)$ is concave up and the intervals on which it is concave down. Identify all inflection points.

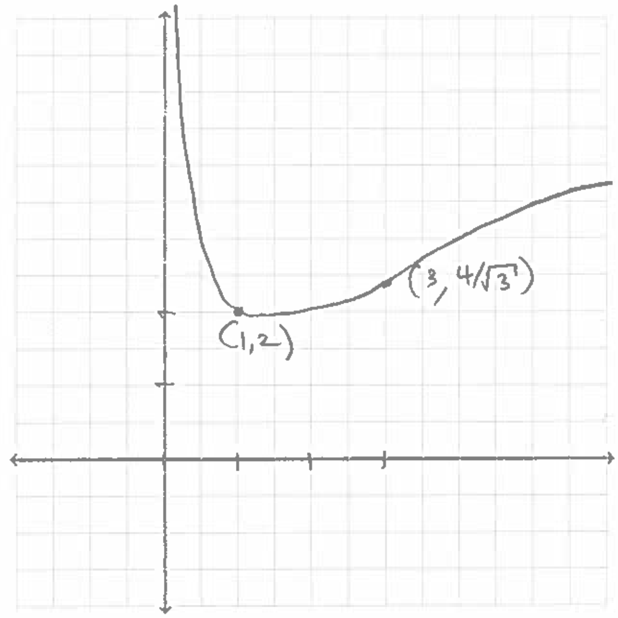
$$f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2}$$
$$= -\frac{1}{4}x^{-5/2}(x-3)$$

$f''(x) = 0$ at $x = 3$

	$(0, 3)$	$(3, \infty)$
sign of f''	$+$	$-$
f	$\cup$ c. up	$\frown$ c. down

$\left(3, \frac{4}{\sqrt{3}}\right)$ inflection pt.

(d) (*Still*: the function f is given by $f(x) = \dfrac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$.) Sketch the graph of $y = f(x)$. Compute all necessary limits and label all points of importance on the graph. (You may use $f(3) = 4/\sqrt{3} \approx 2.3$.)



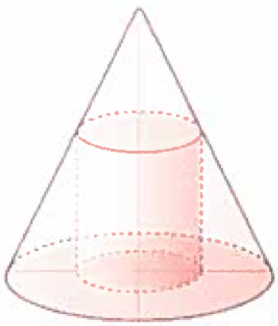
(e) Determine the range of f.

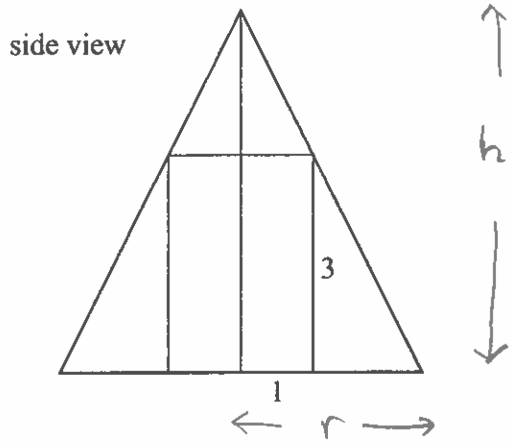
$[2, \infty)$

(f) Is $y = f(x)$ one-to-one on $(0, 2)$?

No. The function is decreasing on $(0,1)$ and increasing on $(1,2)$, so it fails the horizontal line test.

5. A cylinder is inscribed in a cone. The cylinder has height 3 inches and radius of the base 1 inch. Determine the dimensions of such cone with the smallest volume. *Justify all your conclusions*. (*Hint*. Recall that the volume of a cone with radius of the base r and height h is $\frac{1}{3}\pi r^2 h$. Use similar triangles.)





Similar triangles;

$$\frac{r}{h} = \frac{r-1}{3}$$

$$h = \frac{3r}{r-1}$$

$$V = \frac{1}{3}\pi r^2 h = \pi \frac{r^3}{r-1}$$

minimize on $r \geq 1$

$$\frac{dV}{dr} = \pi \left[\frac{3r^2(r-1) - r^3}{(r-1)^2} \right]$$

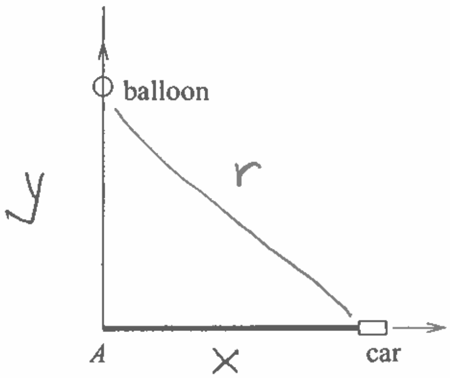
$$= \pi \frac{r^2(2r-3)}{(r-1)^2} = 0 \text{ when } r = 3/2$$

	$(1, 3/2)$	$(3/2, \infty)$
$\frac{dV}{dr}$	$-$	$+$
V	$\searrow$	$\nearrow$

global min at $\underline{\underline{r = 3/2}}$

$$\underline{\underline{h}} = \frac{9/2}{1/2} = \underline{\underline{9}}$$

6. A balloon is rising vertically above a straight road, starting at the point A in the figure. A car is slowly driving on the road *away* from A. At some point in time, the balloon is at 0.4 miles above A, rising at the speed of 5 mph, while the car is 0.3 miles from A, driving at the speed of 30 mph. Determine the speed at which the distance between the car and the balloon is changing at that instance. (*Note*: $0.3^2 = 0.09$, $0.4^2 = 0.16$, $0.5^2 = 0.25$.)



$$r^2 = x^2 + y^2$$

$$\cancel{2} r \frac{dr}{dt} = \cancel{2} x \frac{dx}{dt} + \cancel{2} y \frac{dy}{dt}$$

$$\frac{dr}{dt} = \frac{x}{r}\frac{dx}{dt} + \frac{y}{r}\frac{dy}{dt}$$

When $x = 0.3$, $y = \cancel{0.5}$

$r = \sqrt{x^2+y^2} = \sqrt{0.25} = 0.5$

Plug in: $x = 0.3$, $\frac{dx}{dt} = 30$, $y = 0.4$, $\frac{dy}{dt} = 5$, $r = 0.5$;

$$\frac{dr}{dt} = \frac{3}{5} \cdot 30 + \frac{4}{5} \cdot 5 = \underline{\underline{22}} \quad (\text{mph})$$

(b) At what rate is the area of the triangle determined by the balloon, the car, and point A changing at the same moment?

$$A = \frac{1}{2} xy$$

$$\frac{dA}{dt} = \frac{1}{2}\left(x \frac{dy}{dt} + y \frac{dx}{dt}\right)$$

Plug in: $x = 0.3$, $y = 0.4$, $\frac{dx}{dt} = 30$, $\frac{dy}{dt} = 5$

$$= \frac{1}{2}\left(0.3 \cdot 5 + 0.4 \cdot 30\right)$$

$$= \frac{13.5}{2} = \underline{\underline{\frac{27}{4}}}$$

7. *In all parts of this problem*, the function f is given by $f(x) = \dfrac{x^2+9}{x^2} = 1 + 9x^{-2}$.

(a) Identify the domain, monotonicity and concavity properties of this function and sketch its graph.

Domain: $x \neq 0$

$\lim_{x \to 0} f(x) = \infty$

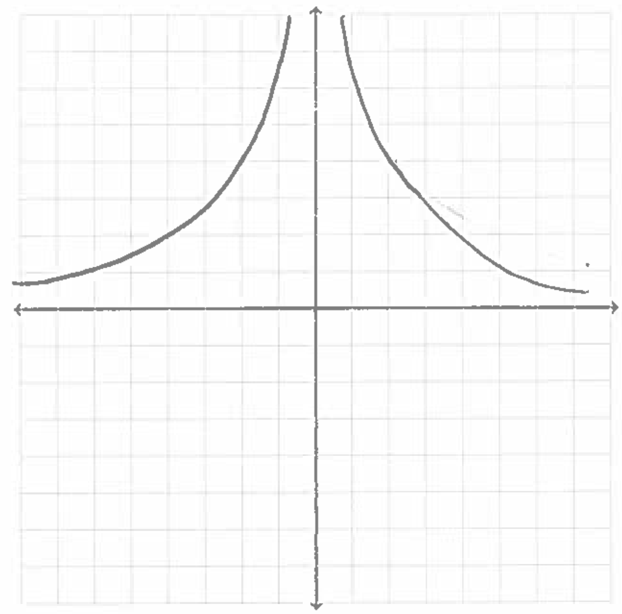
$f'(x) = -18x^{-3}$

$f'(x) > 0$ when $x < 0$

$f'(x) < 0$ when $x > 0$

$f''(x) = 54x^{-4} > 0$

f is always concave up

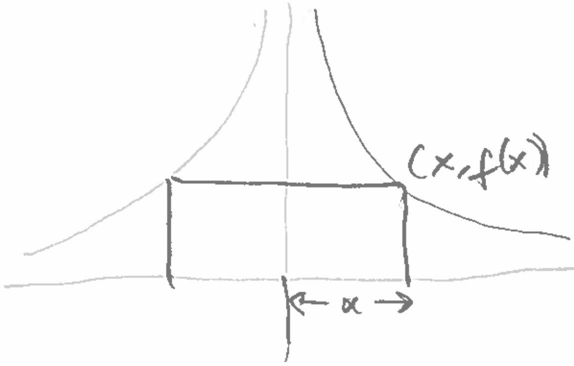


(b) Is this function odd, even, or neither?

f is an <u>even function</u>.

$f(-x) = f(x)$

(c) (*Still*: the function f is given by $f(x) = \dfrac{x^2+9}{x^2} = 1 + 9x^{-2}$.) What is the smallest area of a rectangle with two of its vertices on the x-axis and two of its vertices on the graph of $y = f(x)$?



$x > 0$

$$A = 2x\,f(x) = 2\left(\frac{x^2+9}{x}\right) = 2\left(x + \frac{9}{x}\right)$$

find min. for $x > 0$

$$\frac{dA}{dx} = 2\left(1 - \frac{9}{x^2}\right) \qquad = 0 \text{ for } x = 3$$

	$(0, 3)$	$(3, \infty)$
$\frac{dA}{dx}$	$-$	$+$
A	↘	↗

global min when $x = 3$, and the minimal area is $2 \cdot (3 + 3) = \underline{12}$

8. Provide straightforward, and *fully justified*, answers to the following questions. In each of them, assume that $y = f(x)$ is a continuous function defined for all x, and $f'(x)$ and $f''(x)$ exist and are continuous for all x. (*Note*: assumptions in (a) apply only to (a); the same is true for (b) and (c).)

(a) $f(-2) = 3$, $f(4) = -1$, and $f'(x) < 0$ for all x. How many x-intercepts does f have?

By Rolle, as f' is never zero, f can have at most one x-intercept. But $f(-2)>0$ and $f(4)<0$, so by IVT, f must have at least one x-intercept. Answer: $\boxed{1}$

(b) $f''(x) > 0$ for all x. Is it possible that f is one-to-one? (Either prove that it is not possible or give an example of such a function.)

$\boxed{\text{Yes.}}$ Example: $f(x) = e^x$

$f'(x) = e^x > 0$ for all x, so f is one-to-one by Rolle.

$f''(x) = e^x > 0$

(c) $f(-2) = 5$, $f'(x) \leq 2$ for all x. Is it possible that $f(3) = 20$? (Either prove that it is not possible or give an example of such a function.)

MVT: if $f(3)=20$, then

$$\frac{f(3) - f(-2)}{3 - (-2)} = f'(c) \quad \text{for some } c$$

So $f'(c) = \dfrac{20-5}{5} = 3 > 2$ for some c. $\boxed{\text{NO}}$