Math 21A, Fall 2022.
Oct. 19, 2022.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

1. *In both parts of this problem*, consider the function

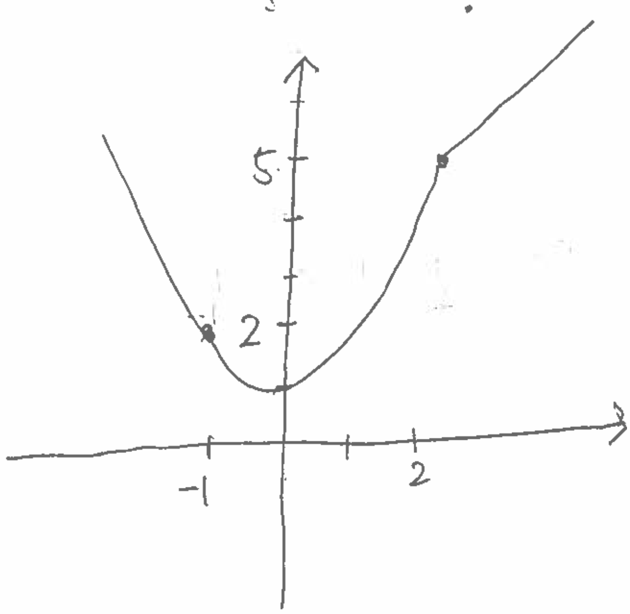
$$f(x) = \begin{cases} -2x & \text{if } x \le -1 \\ ax^2 + b & \text{if } -1 < x < 2 \\ x + 3 & \text{if } x \ge 2 \end{cases}$$

(a) Determine a and b so that the function $y = f(x)$ is continuous for all x. Then sketch its graph and determine its range.

Cont. at -1: $2 = a + b$, $a + b = 2$ $\qquad 3a = 3$

Cont. at 2: $4a + b = 5$ $\qquad \boxed{a = 1,\ b = 1}$

$$f(x) = \begin{cases} -2x & x \le -1 \\ x^2 + 1 & -1 < x < 2 \\ x + 3 & x \ge 2 \end{cases}$$



Range: all $y \ge 1$, that is, the interval $[1, \infty)$.

(b) Assume values of a and b determined in (a). For which x is the function $y = f(x)$ differentiable?

$$f'(x) = \begin{cases} -2 & x < -1 \\ 2x & -1 < x < 2 \\ 1 & x > 2 \end{cases}$$

Diff. at -1: $-2 = -2$, Yes, the left and right derivatives match.

Diff. at 2: $4 \neq 1$, No, the two derivatives do not math.

The function is differentiable everywhere except at $x = 2$.

$$\frac{3x(x-1)}{(x-2)(x+2)}$$

2. Consider the function $f(x) = \dfrac{3(x^2-x)}{x^2-4}$. Determine the domain, intercepts, and vertical and horizontal asymptotes. Determine also any points where the graph of $y = f(x)$ intersects its horizontal asymptote. Then sketch the graph of this function on which all obtained points and asymptotes are clearly marked.

Domain: $x \neq -2, 2$

Intercepts: $(0,0), (1,0)$

Horizontal asymptote: $\lim_{x\to\infty} f(x) = 3$; $y = 3$

Vertical asymptotes:

$\lim_{x\to 2+} f(x) = \infty$ $\quad \dfrac{3(\approx 2)(\approx 1)}{(\text{small}>0)(\approx 4)}$

$\lim_{x\to 2-} f(x) = -\infty$

changes sign

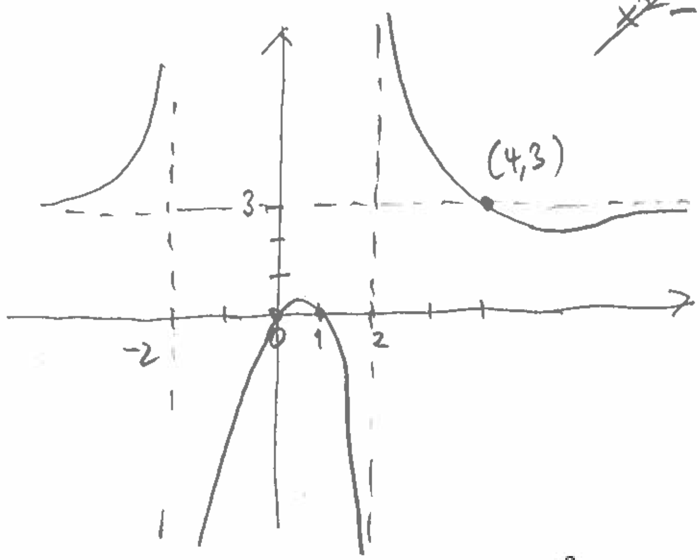
$\lim_{x\to -2+} f(x) = -\infty$ $\quad \dfrac{3(\approx -2)(\approx -3)}{(\approx -4)(\text{small}>0)}$

$\lim_{x\to -2-} f(x) = +\infty$

changes sign

Intersection with h.a. $\quad \dfrac{3(x^2-x)}{x^2-4} = 3$

$\cancel{x^2} - x = \cancel{x^2} - 4 \quad x = 4 \quad (4,3)$



3. Compute the following limits. Give each answer as a finite number, $+\infty$, or $-\infty$.

(a) $\lim_{x\to 0} \dfrac{7x+\sin(3x)}{3x+\sin x + 2x\sqrt{x+1}}$ $\quad /x$, $/x$

$$= \lim_{x\to 0} \frac{7 + \underbrace{\frac{\sin 3x}{3x}}_{\to 1} \cdot 3}{3 + \underbrace{\frac{\sin x}{x}}_{\to 1} + 2\sqrt{x+1}}$$

$$= \frac{7+3}{3+1+2} = \frac{10}{6} = \underline{\underline{\frac{5}{3}}}$$

(b) $\lim_{x\to 2} \dfrac{x^2-4}{\sqrt{x+7}-3}$ $\quad \cdot \dfrac{\sqrt{x+7}+3}{\sqrt{x+7}+3}$

$$= \lim_{x\to 2} \frac{(x^2-4)(\sqrt{x+7}+3)}{x+7-9}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)(\sqrt{x+7}+3)}{\cancel{x-2}} = 4 \cdot 6 = \underline{\underline{24}}$$

4. *In all parts of this problem,*

$$f(x) = \frac{x^2-4}{2x+1}.$$

(a) Determine $L = \lim_{x \to 1} f(x)$.

$L = -1$

(b) For a given $\epsilon > 0$, determine a $\delta > 0$ so that $|x-1| < \delta$ will guarantee that $|f(x) - L| < \epsilon$.

$$\left| \frac{x^2-4}{2x+1} + 1 \right| < \varepsilon$$

$$\left| \frac{x^2-4+2x+1}{2x+1} \right| < \varepsilon$$

$$\left| \frac{x^2+2x-3}{2x+1} \right| < \varepsilon$$

$$\frac{|x-1|\,|x+3|}{|2x+1|} < \varepsilon$$

Assume $|x-1| < 1$

$-1 < x-1 < 1$

$0 < x < 2$

$\underline{3 < x+3 < 5}$

$\underline{1 < 2x+1} < 5$

$$5|x-1| < \varepsilon$$

$$|x-1| < \frac{\varepsilon}{5}$$

$$\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$$

5. *In both parts of this problem,* $f(x) = 4\sqrt{x} + x - 3$, and assume that $x > 0$.
(a) A line is tangent to the graph of $y = f(x)$ and parallel to the line $y = 2x + 17$. Determine the equation of this line (in the slope-intercept form).

$$f'(x) = \frac{2}{\sqrt{x}} + 1$$

Parallel to $y = 2x + 17$: $f'(x) = 2$

$\frac{2}{\sqrt{x}} + 1 = 2$, $\sqrt{x} = 2$, $x = 4$, $f(4) = 9$

Point : $(4, 9)$

Tangent line at this point:

$y - 9 = 2(x - 4)$

$\underline{y = 2x + 1}$

(b) Let $g(x) = x^3$. Do the graphs of $y = f(x)$ and $y = g(x)$ intersect for some $x \geq 1$? Clearly explain your answer.

$h(x) = f(x) - g(x)$

$h(1) = f(1) - g(1) = 2 - 1 = 1 > 0$

$h(4) = f(4) - g(4) = 9 - 64 < 0$

By IVT, there is an x in $[1, 4]$ at which $h(x) = 0$, that is, $f(x) = g(x)$. <u>Yes</u>.