Math 21A, Winter 2024.
Feb. 7, 2024.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. *In both parts of this problem*, consider the function

$$f(x) = \begin{cases} x+2 & \text{if } x \le 0 \\ ax+b & \text{if } 0 < x < 1 \\ 1-x^2 & \text{if } x \ge 1 \end{cases}$$

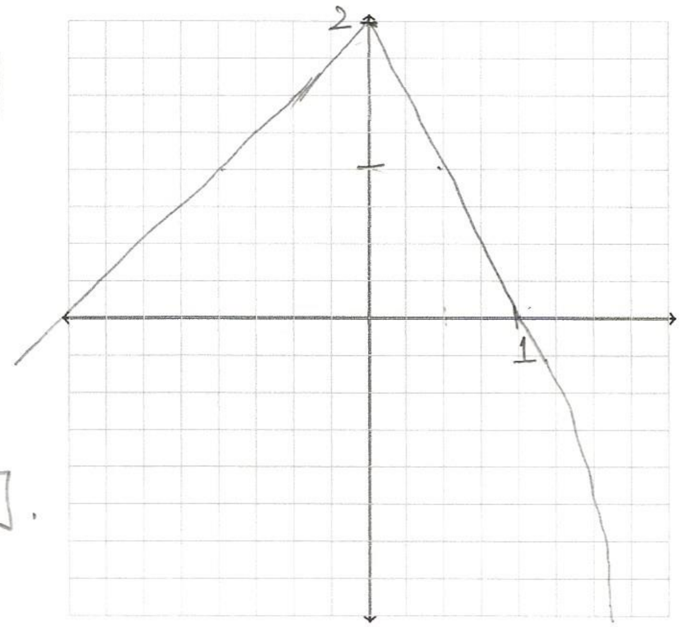
(a) Determine a and b so that the function $y = f(x)$ is continuous for all x. Then sketch its graph and determine its range.

Cont. at $x=0$: $2=b$ $\boxed{b=2}$

Cont at $x=1$: $a+b=0$, $\boxed{a=-2}$

$$f(x) = \begin{cases} x+2 & x \le 0. \\ -2x+2 & 0<x<1 \\ 1-x^2 & x \ge 1 \end{cases}$$

Range: all $y \le 2$, that is, the interval $(-\infty, 2]$.



(b) Assume values of a and b determined in (a). For which x is the function $y = f(x)$ differentiable?

$$f'(x) = \begin{cases} 1 & x<0 \\ -2 & 0<x<1 \\ -2x & x>1 \end{cases}$$

Answer. The function is differentiable everywhere except at $x=0$.

Diff at 0 : $1 \ne -2$. No, the left and right derivatives do not match.

Diff at 1 : $-2 = -2$, Yes, the two derivatives do match.

$$= \frac{2(x-2)(x+2)}{x(x+4)}$$

2. Consider the function $f(x) = \dfrac{2(x^2-4)}{x^2+4x}$. Determine the domain, intercepts, and vertical and horizontal asymptotes. Determine also any points where the graph of $y = f(x)$ intersects its horizontal asymptote. Then sketch the graph of this function on which all obtained points and asymptotes are clearly marked.

Domain: $x \neq 0, -4$

Intercepts: $(-2,0)$ $(2,0)$

Horizontal asymptote:

$\lim_{x\to\infty} f(x) = 2$; $y = 2$

Vertical asymptotes

$\lim_{x\to 0+} f(x) = -\infty$

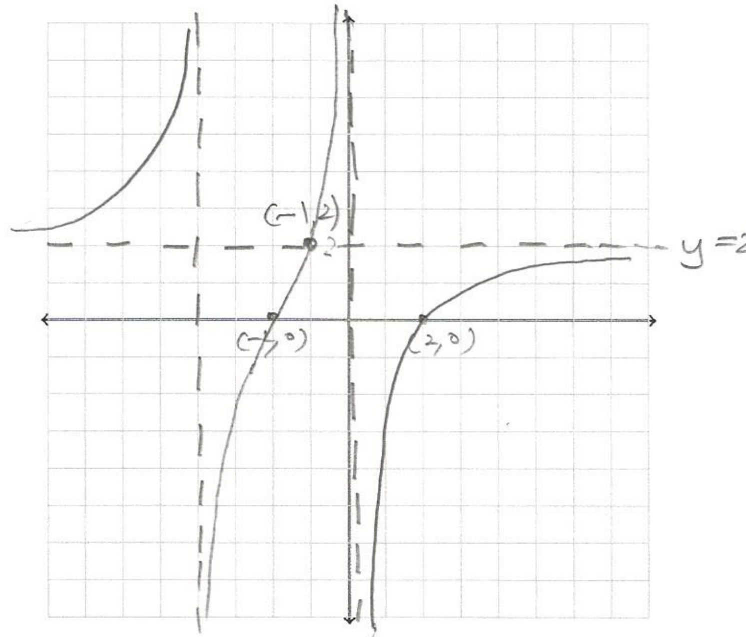
$$\frac{3(\approx -2)(\approx 2)}{(\text{small} > 0)(\approx 4)}$$

$\lim_{x\to 0-} f(x) = +\infty$ (changes sign)

$\lim_{x\to -4+} f(x) = -\infty$

$$\frac{2\cdot(\approx -6)(\approx -2)}{(\approx -4)(\text{small} > 0)}$$

$\lim_{x\to -4-} f(x) = +\infty$ (changes sign)



Intersections with h.a.:

$$\frac{2(x^2-4)}{x^2+4x} = 2$$

$$\cancel{x^2} - 4 = \cancel{x^2} + 4x$$

$$x = -1$$

$(-1,2)$

3. Compute the following limits. Give each answer as a finite number, $+\infty$, or $-\infty$.

(a) $\lim_{x\to 3} \frac{x-3}{\sqrt{x^2-5}-2}$ $\cdot \frac{\sqrt{x^2-5}+2}{\sqrt{x^2-5}+2}$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x^2-5}+2)}{x^2-5-4} \qquad = x^2-9$$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}(\sqrt{x^2-5}+2)}{\cancel{(x-3)}(x+3)} = \frac{\sqrt{4}+2}{3+3} = \frac{2}{3}$$

(b) $\lim_{x\to 0} \frac{x\sqrt{4+x^2}+\sin(3x)}{2x-\sin(3x)}$ $\begin{matrix} /x \\ /x \end{matrix}$

$$= \lim_{x\to 0} \frac{\sqrt{4+x^2} + \underbrace{\frac{\sin(3x)}{3x}}_{\to 1}\cdot 3}{2 - \underbrace{\frac{\sin(3x)}{3x}}_{\to 1}\cdot 3}$$

$$= \frac{2+3}{2-3} = -5$$

(c) $\lim_{x\to\infty} \frac{5x+\sin(3x)}{2x-\sin(3x)}$ $\begin{matrix} /x \\ /x \end{matrix}$

$\to 0$ (as $-1 \le \sin(3x) \le 1$ and so $-\frac{1}{x} \le \frac{\sin(3x)}{x} \le \frac{1}{x}$)

$$= \lim_{x\to\infty} \frac{5 + \frac{\sin(3x)}{x}}{2 - \underbrace{\frac{\sin(3x)}{x}}_{\to 0}}$$

$$= \frac{5}{2}$$

4. *In both parts of this problem,* $f(x) = x + \frac{2}{x} + 1$. $= x + 2x^{-1} + 1$

(a) There are two lines that are tangent to the graph of $y = f(x)$ and parallel to the line $y = \frac{1}{2}x + 24$. Determine the equations of of both of these lines (in the slope-intercept form).

$f'(x) = 1 - 2x^{-2}$

Parallel to $y = \frac{1}{2}x + 24$; $f'(x) = \frac{1}{2}$

$1 - 2x^{-2} = \frac{1}{2}$ $\quad x^{-2} = \frac{1}{4}$ $\quad x^2 = 4$

$x = -2, 2$

Points: $(-2,-2)$, $(2,4)$

Tangent lines: $y + 2 = \frac{1}{2}(x+2)$ $\quad \underline{\underline{y = \frac{1}{2}x - 1}}$

$y - 4 = \frac{1}{2}(x-2)$ $\quad \underline{\underline{y = \frac{1}{2}x + 3}}$

(b) Let $g(x) = x^4$. Do the graphs of $y = f(x)$ and $y = g(x)$ intersect for some $x \geq 1$? Clearly explain your answer.

$h(x) = f(x) - g(x)$ is a continuous function.

$h(1) = f(1) - g(1) = 4 - 1 = 3 > 0$

$h(2) = f(2) - g(2) = 4 - 16 = -12 < 0$

By IVT, there is an x in $[1,2]$ at which $h(x) = 0$, that is, $f(x) = g(x)$. Yes.