Math 21A, Fall 2022.
Nov. 16, 2022.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

1. Compute the derivatives of the following two functions. *Do not simplify!* (Please be careful; you will receive little or no credit if you make a differentiation mistake, even a small one.)

(a) $y = \ln(x + \sqrt{x})$ $= \ln(x + x^{1/2})$

$$y' = \frac{1}{x+\sqrt{x}} \cdot \left(1 + \frac{1}{2} x^{-1/2}\right)$$

(b) $y = \arctan(x \cdot e^{3x})$

$$y' = \frac{1}{1+(x e^{3x})^2} \cdot \left(e^{3x} + x \cdot e^{3x} \cdot 3\right)$$

2. Find the equation of the tangent line to the curve $x^2y + \sin(xy) + 2y^2 = 2$ at the point $(0, 1)$. You may leave the equation of the line in the point-slope form. (Please be careful; you will receive little or no credit if you make a differentiation mistake, even a small one.)

$$2xy + x^2y' + \cos(xy)\,(y + xy') + 4yy' = 0$$

Plug in $x=0$, $y=1$;

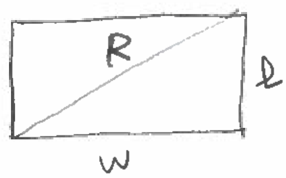
$$1 + 4y' = 0 \qquad y' = -1/4 \leftarrow \text{slope}$$

Line:

$$y - 1 = -\frac{1}{4}x$$

$$\underline{\underline{y = -\frac{1}{4}x + 1}}$$

3. The length ℓ of a rectangle is decreasing at the rate of 2m/sec while the width w is increasing at the rate of 5m/sec. When $\ell = 4$m and $w = 3$m, find the rates of change of (a) the area and (b) the length of the diagonals of the rectangle.



Area A: $A = w \cdot \ell$

$$\frac{dA}{dt} = w \cdot \frac{d\ell}{dt} + \ell \cdot \frac{dw}{dt}$$

Plug in $w = 3$, $\ell = 4$, $\frac{d\ell}{dt} = -2$, $\frac{dw}{dt} = 5$:

$$\frac{dA}{dt} = 3 \cdot (-2) + 4 \cdot 5 = 14$$

Diagonal R: $R = \sqrt{w^2 + \ell^2}$

$$\frac{dR}{dt} = \frac{1}{\cancel{2}\sqrt{w^2+\ell^2}} \left(\cancel{2}w \frac{dw}{dt} + \cancel{2}\ell \frac{d\ell}{dt} \right)$$

Plug in as above:

$$\frac{dR}{dt} = \frac{1}{\sqrt{25}} \left(3 \cdot 5 + 4 \cdot (-2) \right) = \frac{7}{5}$$

4. *In all parts of this problem*, consider the function $f(x) = \dfrac{x^2+4}{x}$ on the domain $D = [1,4]$.

(a) Determine the global maximum and minimum of $y = f(x)$ on D, and its range on D. (Note: global maximum and minimum are the same as absolute maximum and minimum.)

$$f'(x) = \frac{2x \cdot x - (x^2+4)}{x^2} = \frac{x^2-4}{x^2} = \frac{(x-2)(x+2)}{x^2}$$

$f'(x) = 0$ for $x = -2, 2$, so the only critical number $\underbrace{\text{is}}_{\text{in } (1,4)}$ $x = 2$.

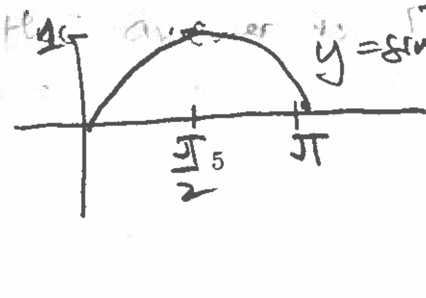
x	$f(x)$
1	5
2	4 ← min
4	5 ← max

The global min. is at $x=2$, with value 4, and global max. is at $x = 1, 4$ with value 5.

Range: $[4, 5]$.

(b) Let $g(x) = \sin(\pi f(x))$. Determine the range of the function $y = g(x)$ on D.

This is the same as the range of sin on $[4\pi, 5\pi]$, which by periodicity of sin is the same as the range on $[0, \pi]$, which is $\underline{[0, 1]}$.



5. Let $f(t) = 3t - \sin(2t)$. The position s at time $t \geq 0$ of a particle moving along a coordinate line is given by $s = f(t)$.
(a) Find the smallest time $t \geq 0$ at which the acceleration is 4. Find the velocity at that time.

$$f'(t) = 3 - 2\cos(2t)$$

$$f''(t) = 4\sin(2t)$$

smallest

$f''(t) = 4$ when $\sin(2t) = 1$, $2t = \frac{\pi}{2}$, $\underline{\underline{t = \frac{\pi}{4}}}$.

At this time $t = \frac{\pi}{4}$,

$$f'\left(\frac{\pi}{4}\right) = 3 - 2\cos\left(\frac{\pi}{2}\right) = \underline{\underline{3}}$$

(b) At how many times $t \geq 0$ is the particle positioned at $s = 1$?

$f'(t) = 3 - 2\cos(2t) \geq 1$
is <u>never</u> zero, so there is at most one time t at which $f(t) = 1$, by Rolle's thm.

$f(0) = 0$

$f\left(\frac{\pi}{2}\right) = 3\frac{\pi}{2} - \sin\pi = \frac{3\pi}{2} > 1$

By IVT, there is at least one t in $\left(0, \frac{\pi}{2}\right)$ at which $f(t) = 1$.

<u>Answer</u>: at a <u>single</u> time.