Math 21A, Winter 2024
Feb. 28, 2024.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

1. Compute the derivatives of the following two functions. *Do not simplify!* (Please be careful; you will receive little or no credit if you make a differentiation mistake, even a small one.)

(a) $y = \ln(e^x + 1)$

$$y' = \frac{e^x}{e^x+1}$$

(b) $y = \dfrac{\arcsin x}{x^3 + x}$

$$y' = \frac{\frac{1}{\sqrt{1-x^2}}(x^3+x) - \arcsin x \cdot (3x^2+1)}{(x^3+x)^2}$$

2. Find the equation of the tangent line to the curve $x^4 + y^3 + 3\arctan(xy) - y = 0$ at the point $(0,1)$. You may leave the equation of the line in the point-slope form. (Please be careful; you will receive little or no credit if you make a differentiation mistake, even a small one.)

$$4x^3 + 3y^2 \cdot y' + 3 \cdot \frac{1}{1+(xy)^2}(y + xy') - y' = 0$$

Plug in $x=0, y=1$:

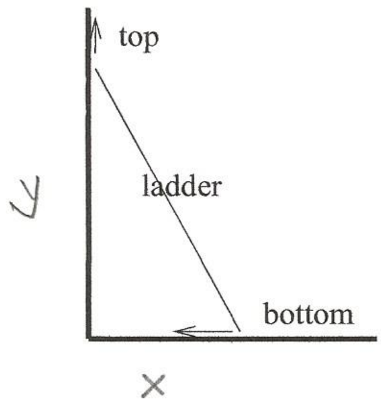
$$3y' + 3 - y' = 0$$

$$2y' = -3, \qquad y' = -\frac{3}{2} \leftarrow \text{slope}$$

Line: $y - 1 = -\frac{3}{2}x$

$$y = -\frac{3}{2}x + 1$$

3. A 13-foot ladder is leaning against the wall. At one instant, the top of the ladder is 12 feet from the ground and is pulled up the wall at the rate of 10 feet per minute. At what rate is the bottom of the ladder moving towards the wall at that instant? (You may need this: $13^2 = 169$, $12^2 = 144$, $169 - 144 = 25$.)



$$x^2 + y^2 = 13^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = \frac{-y}{x} \frac{dy}{dt}$$

When $y = 12$, $x = \sqrt{13^2 - 12^2} = \sqrt{25} = 5$.

Plug in $x = 5, y = 12, \frac{dy}{dt} = 10$ to get

$$\frac{dx}{dt} = -\frac{12}{5} \cdot 10 = -24 \quad (\text{ft}/\text{min})$$

4. *In all parts of this problem*, consider the function $f(x) = 4\sqrt{x} - x$ on the domain $D = [0, 9]$.

(a) Determine the global maximum and minimum of $y = f(x)$ on D, and its range on D. (Note: global maximum and minimum are the same as absolute maximum and minimum.)

$$f'(x) = 4 \cdot \frac{1}{2} x^{-1/2} - 1 = \frac{2}{\sqrt{x}} - 1$$

$f'(x) = 0$ when $\frac{2}{\sqrt{x}} = 1$, $\sqrt{x} = 2$, $x = 4$

(undefined at 0, but 0 is already a boundary pt.)

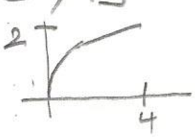
x	$f(x)$	
0	0	$\leftarrow$ global min
4	4	$\leftarrow$ global max
9	3	

Range: $[0, 4]$

(b) Explain why the function $g(x) = \sqrt{f(x)}$ is defined on the entire domain D. Then determine the range of the function $y = g(x)$ on D.

$f(x) \geq 0$ on D, so $\sqrt{f(x)}$ is defined on D

Range: Square roots of numbers in $[0, 4]$

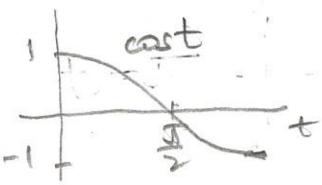
(i.e. the range of $y = \sqrt{x}$ on $[0, 4]$; )

so the range of g is $[0, 2]$

5. Let $f(t) = 3t + \sin t - 1$. The position s at time $t \geq 0$ of a particle moving along a coordinate line is given by $s = f(t)$.
(a) Find the smallest time $t \geq 0$ at which the velocity is 2. Find the acceleration at that time.

$v = f'(t) = 3 + \cos t$

$3 + \cos t = 2 \qquad \cos t = -1,$



The smallest sol.: $\underline{t = \pi}$

$a = f''(t) = -\sin t$

At $t = \pi$, the acc. is $f''(\pi) = -\sin\pi = 0$.

(b) At how many times $t \geq 0$ is the particle positioned at $s = 0$?

$f'(t) = 3 + \cos t \geq 2$ is <u>never zero</u>

By Rolle's thm., there is <u>at most one</u> time t at which $f(t) = 0$.

$f(0) = -1 < 0$

$f(\pi) = 3\pi - 1 > 0$

By IVT, there is <u>at least one</u> time t on $[0, \pi]$ at which $f(t) = 0$.

<u>Answer:</u> At a <u>single time</u>.