Week 10 Lectures

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11.3–11.5. Polar coordinates

A point in the plane, say $(4, 2)$ can also be described by its distance r from the origin and the angle θ between the ray from the origin through the point and the positive part of the x-axis. We call (r, θ) the *polar coordinates*.

In this case,

$$r = \sqrt{x^2 + y^2} = \sqrt{20} = 2\sqrt{5}, \quad \theta = \arctan \frac{1}{2} \approx 0.46 \approx 27^\circ.$$

Note that $(r, \theta + 2\pi)$ represents the same point as (r, θ) in polar coordinates. The point $(r, -\theta)$ represents $(r, 2\pi - \theta)$.

We sometimes use *the negative r convention*: if $r < 0$, we *define* $(r, \theta) = (-r, \theta + \pi)$. Unless explicitly specified, we will not use this convention but will consider as legitimate only points with positive r-coordinate.

Connection between Cartesian and polar coordinates: let (x, y) be the Cartesian and (r, θ) be the polar coordinates of the same point, with $r \geq 0$ and θ in $[0, 2\pi)$. Then

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$

Example. Polar coordinates on $(1, 1)$: $r = \sqrt{2}$, $\theta = \frac{\pi}{4}$.
Polar coordinates on $(-1, -1)$: $r = \sqrt{2}$, $\theta = \frac{5\pi}{4}$.

Examples.
Graph the following curves in polar coordinates:

(1) $r = 1$

(2) $r = 1 - \cos\theta$.

This is the *Cardioid*. This is the curve traced by a point on the perimeter of a circle of diameter 1 that is rolling around a fixed circle of the same diameter. Its equation in the Cartesian coordinates is given by

$$r^2 = r - r\cos\theta$$

$$x^2 + y^2 = \sqrt{x^2 + y^2} - x,$$

a much more complicated equation.

(3) $r = \theta$

This is the *Archimedian spiral*.

(4) $r = \sin 2\theta$.

Explain the effect of the negative r convention.

Without the convention, this curve does not exist in the 2nd ($\pi/2 < \theta < \pi$) and 4th ($3\pi/2 < \theta < 2\pi$) quadrants. The negative r convention produced two new “leafs” in those quadrants.

Compute the intersections with the circle $r = 1/2$ in the first quadrant.

To solve, we write

$$\sin 2\theta = 1/2$$

$$2\theta = \frac{\pi}{6} + 2k\pi, \quad 2\theta = \frac{5\pi}{6} + 2k\pi$$

$$\theta = \frac{\pi}{12}, \frac{\pi}{12} + \pi, \quad \theta = \frac{5\pi}{12}, \theta = \frac{5\pi}{12} + \pi$$

$$\theta = \frac{\pi}{12}, \quad \theta = \frac{5\pi}{12} \quad \text{in the 1st quadrant } (0 \leq \theta \leq \pi/2)$$

11.3–11.5. Polar coordinates

The polar equation is a special case of a parametric representation of a curve: if $r = f(\theta)$, then using

$$x = r\cos\theta, \quad y = r\sin\theta$$

we get parametric representation

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

In principle, all calculus problems are thus solved by the methods developed for parametric curves, but often it is better to start from first principles.

Area in polar coordinates.

Assume that we have a curve given in polar coordinates by $r = f(\theta)$ for θ in $[\alpha, \beta]$. We want to compute the area of the sector traced by the radial line segments, at angles from α to β, from the origin to the curve.

A sector between θ and $\theta + \Delta\theta$ has area approximated by that of a circular segment with constant radius $r = f(\theta)$, whose area is

$$\frac{1}{2}r^2\Delta\theta = \frac{1}{2}f(\theta)^2\Delta\theta$$

So,

$$\text{Area} = \int_\alpha^\beta \frac{1}{2}r^2\,d\theta = \int_\alpha^\beta \frac{1}{2}f(\theta)^2\,d\theta$$

Example. Area of the cardioid.

By the formula

$$\begin{aligned}\text{Area} &= \int_0^{2\pi} \frac{1}{2} f(\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)\, d\theta \\ &= \frac{1}{2}\cdot\frac{3}{2}\cdot 2\pi = \frac{3\pi}{2}\end{aligned}$$

Example. Area between curves $r = 1 - \cos\theta$ and $r = \cos\theta$. (By default: no negative r convention!)

We solve

$$1 - \cos\theta = \cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

The region consists of two symmetric pieces, and each needs to be divided into two pieces.

In fact, $r = \cos\theta$ is a circle, as $r^2 = r\cos\theta$ gives

$$x^2 + y^2 = x$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

By the formula

$$\begin{aligned}
\text{Area} &= 2 \cdot \frac{1}{2}\left[\int_0^{\pi/3} (1-\cos\theta)^2 \, d\theta + \int_{\pi/3}^{\pi/2} \cos^2\theta d\theta\right] \\
&= \int_0^{\pi/3} (1-2\cos\theta) \, d\theta + \int_0^{\pi/2} \cos^2\theta d\theta \\
&= \int_0^{\pi/3} (1-2\cos\theta) \, d\theta + \frac{1}{2}\int_0^{\pi/2} (1+\cos 2\theta) d\theta \\
&= \frac{\pi}{3} - 2\sin\theta\Big|_0^{\pi/3} + \frac{\pi}{4} + \frac{1}{4}\sin 2\theta\Big|_0^{\pi/2} \\
&= \frac{7\pi}{12} - \sqrt{3} \approx 0.1005
\end{aligned}$$

Arc length in polar coordinates.
Assume that we have a curve given in polar coordinates by $r = f(\theta)$ for θ in $[\alpha, \beta]$. We want to compute the arc length between angles α and β. This time we use the parametric interpretation: $x = r\cos\theta, \quad y = r\sin\theta$, so

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$
$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta$$
$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta$$

so that (do the algebra in your head!)

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$
$$= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

So,

$$\text{Arc Length} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example. The length of the cardioid.

$$\begin{aligned}
r &= 1 - \cos\theta \\
\frac{dr}{d\theta} &= \sin\theta \\
r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1-\cos\theta)^2 + \sin^2\theta \\
&= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta \\
&= 2 - 2\cos\theta \\
&= 4\sin^2\frac{\theta}{2}
\end{aligned}$$

So,

$$\text{Arc Length} = \int_0^{2\pi} 2\sin\frac{\theta}{2}\,d\theta$$
$$= -4\cos\frac{\theta}{2}\Big|_0^{2\pi} = 8$$

Example. Compute the slope of the tangent to $r = \sin 2\theta$ at $\theta = \frac{\pi}{6}$.

$$x = \sin 2\theta \, \cos\theta$$
$$y = \sin 2\theta \, \sin\theta$$

$$\frac{dx}{d\theta} = 2\cos 2\theta \, \cos\theta - \sin 2\theta \, \sin\theta$$
$$\frac{dy}{d\theta} = 2\cos 2\theta \, \sin\theta + \sin 2\theta \, \cos\theta$$

At $\theta = \frac{\pi}{6}$, we have $\frac{dx}{d\theta} = \frac{\sqrt{3}}{4}$, $\frac{dy}{d\theta} = \frac{5}{4}$ and

$$\text{slope} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5}{\sqrt{3}}$$

Example. *Leminiscate*. This curve is given by

$$r = \sqrt{\cos 2\theta}$$

This is the curve that traces points that have constant product of distances from two foci; in this case, the foci are at $\pm 1/\sqrt{2}$, and the product of the two distances is $1/2$:

$$\begin{aligned}
&d((x,y),(\tfrac{1}{\sqrt{2}},0)) \cdot d((x,y),(-\tfrac{1}{\sqrt{2}},0)) = \frac{1}{2} \\
&((x-\tfrac{1}{\sqrt{2}})^2 + y^2) \cdot ((x+\tfrac{1}{\sqrt{2}})^2 + y^2) = \frac{1}{4} \\
&(x^2 - \sqrt{2}x + \tfrac{1}{2} + y^2) \cdot (x^2 + \sqrt{2}x + \tfrac{1}{2} + y^2) = \frac{1}{4} \\
&(x^2+y^2)^2 = x^2 - y^2 \\
&r^4 = r^2(\cos^2\theta - \sin^2\theta) \\
&r^2 = \cos 2\theta
\end{aligned}$$

The curve is given by

$$r = \sqrt{\cos 2\theta}$$

Compute the area enclosed by the curve and write the integral for its arc length. We compute

$$\frac{dr}{d\theta} = \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}$$

So

$$\text{Area} = 4 \cdot \frac{1}{2} \cdot \int_0^{\pi/4} \cos 2\theta \, d\theta = \sin 2\theta \Big|_0^{\pi/4} = 1$$

$$\text{Arc Length} = 4\int_0^{\pi/4} \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$
$$= 4\int_0^{\pi/4} \frac{1}{\sqrt{\cos 2\theta}}\, d\theta \approx 5.2441$$

The integral is not solvable exactly; it is called an *elliptic integral*.

Optional fun problem.

A good strategy to run away? You are the captain of a US Coast Guard speedboat, patrolling the Pacific coast on a foggy day. All of a sudden, the fog lifts and you observe what is obviously a smuggling boat a distance d away. Unfortunately, the fog instantly descends and the visibility again becomes zero. Here are the assumptions you may make. First, the speed of the smuggling boat is known to be 1 (in appropriate units) and the speed of your boat is $v > 1$. Second, the smuggling boat will move in a straight line in an unknown direction from where it was spotted. Can you move your boat on a curve *guaranteed* to catch the smugglers? (You may suppose that the boats are points and that for the smugglers to be caught the two boats have to be at the same point at the same time.)

Solution. Set the coordinate system so that initially the smugglers are at the origin and the Guard is at $(d, 0)$. Move the Guard boat toward the origin until it is at the point where it would meet the smugglers if they proceeded to move on the positive x axis. This is the point $(d/(v+1), 0)$. From now on, i.e. $t \geq t_0 = d/(v+1)$, move the Guard on the curve

$$(x(t), y(t)) = (t \cdot \cos\theta, t \cdot \sin\theta),$$

where $\theta = \theta(t)$ is the angle to be determined, but $\theta(t_0) = 0$. Note that this means that the Guard is at distance t from the origin at time t, which is true for the smugglers as well! Therefore, if it is possible for the Guard to make θ sweep all angles in $[0, 2\pi)$, we are done.

Now this becomes a calculus problem. First we compute $x'(t)$, $y'(t)$ and then, after a short calculation, obtain

$$x'(t)^2 + y'(t)^2 = 1 + t^2(\theta')^2.$$

As the derivative of the arc length equals, on the one hand, $\sqrt{x'(t)^2 + y'(t)^2}$, and, on the other hand, v, we get

$$v^2 = 1 + t^2(\theta')^2.$$

Thus

$$\theta' = \sqrt{v^2 - 1}/t$$

and

$$\theta = \sqrt{v^2 - 1}\log(t/t_0).$$

The Guard moves on the spiral, given in polar coordinates by

$$r = \frac{d}{v+1} \cdot \exp\left(\frac{\theta}{\sqrt{v^2-1}}\right).$$