Week 2 Lectures

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4.8. Antiderivatives

If F and f are two functions defined on some open interval, and $F'(x) = f(x)$ for all x, then we call F the *antiderivative* (also known as *indefinite integral*) of f. We write

$$\int f(x)\,dx = F(x) + C,$$

emphasizing that we can add a constant C to F without affecting its derivative.

Example. As $(\frac{1}{2}\sin 2x)' = \cos(2x)$, $F(x) = \frac{1}{2}\sin 2x$ is an antiderivative of $\cos(2x)$.

4.8. Antiderivatives

Theorem

If F_1 and F_2 are two antiderivatives of f, they differ by a constant; that is $F_1' = f$ and $F_2' = f$ implies that, for some constant C, $F_2(x) = F_1(x) + C$ for all x.

Proof.

As $(F_1 - F_2)' = 0$, $F_1 - F_2$ is a constant function. □

Example. It follows that *all* antiderivatives of $\cos(2x)$ are $\frac{1}{2}\sin 2x + C$.

Every derivative formula is an antiderivative formula (below, k is a constant).

function	antiderivative
x^n	$\frac{1}{n+1}x^{n+1} + C$, if $n \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\sin(kx)$	$-\frac{1}{k}\cos(kx) + C$
$\cos(kx)$	$\frac{1}{k}\sin(kx) + C$
$\sec^2(kx) = \frac{1}{\cos^2(kx)}$	$\frac{1}{k}\tan(kx) + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$
$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\arcsin(kx) + C$
$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\arctan(kx) + C$

4.8. Antiderivatives

If F is the antiderivative of f, we write

$$F(x) = \int f(x)\,dx,$$

which is where the phrase indefinite integral comes from. The reason for similarity to the definite integral in name and notation will be revealed soon.

This notation is due to Leibniz (1686).

Gottfried Wilhelm von Leibniz (1646–1716) was a German mathematician and philosopher.

Example. We have

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C,$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + C,$$

We have two different answers. Something wrong?

Example. We have

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$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + C,$$

We have two different answers. Something wrong?

No! The two functions differ by a constant:
$\arcsin x = -\arccos x + \frac{\pi}{2}$

Because of analogous properties for derivatives:

Theorem

For a constant c,

$$\int cf(x)\,dx = c\int f(x)\,dx,$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example.

$$\int (x + 5\sin x)\,dx = \frac{1}{2}x^2 - 5\cos x + C$$

$$\int (2x^{10} + 3\sqrt{x})\,dx = \int (2x^{10} + 3x^{1/2})\,dx = \frac{2}{11}x^{11} + 2x^{3/2} + C$$

5.4. The fundamental theorem of calculus

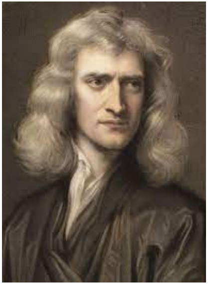
Theorem (First Fundamental Theorem of Calculus)

Let f be a continuous function on an open interval containing $[a, b]$. For x in $[a, b]$, let

$$G(x) = \int_a^x f(t)\,dt.$$

Then G is differentiable on $[a, b]$ and its derivative is f. That is $G'(x) = f(x)$.

This theorem was proved by Isaac Newton at the end of 17th century. Isaac Newton (1643–1727) was an English mathematician, physicist, astronomer, and theologian. He is one of the most influential scientists of all time.



Theorem (Mean Value Theorem, on the way to proving FFTC)

Assume f is continuous on $[a, b]$. Then there exist a $c \in [a, b]$ so that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Proof.

If m is the minimum, and M is the maximum, of f on $[a, b]$,

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

that is, the average value of f is between m and M:

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M.$$

By the IVT, any number between m and M is a value of f. □

5.4. The fundamental theorem of calculus

Proof of FFTC.

Recall that $G(x) = \int_a^x f(t)\,dt$. We need to show that $G'(x) = f(x)$. We will do so by definition of the derivative. Note that

$$G(x+h) = \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$$

and so

$$\frac{G(x+h) - G(x)}{h} = \frac{\int_x^{x+h} f(t)\,dt}{h} = f(c)$$

by MVT, for some c between x and $x+h$.

As $h \to 0$, $c \to x$ and, by continuity, $f(c) \to f(x)$.

So, by the limit definition of the derivative, $G'(x) = f(x)$. □

Theorem (Second Fundamental Theorem of Calculus)

Let f be a continuous function on an open interval containing $[a, b]$, and assume that F is any antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof of SFTC.

By FFTC, the function $G(x) = \int_a^x f(t)\,dt$ is an antiderivative of f. So is $F(x)$. It follows that G and F differ by a constant:

$$\int_a^x f(t)\,dt = F(x) + C.$$

Plug in $x = a$ to get $C = -F(a)$. Plug in $x = b$ to get:

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

□

Example. Compute $\int_1^4 x^2\,dx$

Here $f(x) = x^2$, so its antiderivative $F(x) = \frac{x^3}{3}$. The SFTC says that

$$\int_1^4 x^2\,dx = F(4) - F(1)$$

We often summarize this in shorhand notation:

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_{x=1}^{x=4} = \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1}{3} = 21$$

Example. Compute

$$\int_0^{\pi/2} \sin x \, dx$$

Example. Compute

$$\int_0^{\pi/2} \sin x \, dx$$

Answer:

$$\int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_{x=0}^{x=\pi/2} = 1.$$

Example. (a) Compute

$$\int_1^2 \frac{1}{x}\,dx$$

(b) Can you apply FTC to

$$\int_0^2 \frac{1}{x}\,dx\ ?$$

Example. (a) Compute

$$\int_1^2 \frac{1}{x}\,dx$$

(b) Can you apply FTC to

$$\int_0^2 \frac{1}{x}\,dx \text{ ?}$$

Answer to (a):

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big|_1^2 = \ln 2.$$

Answer to (b): no, because the function $f(x) = 1/x$ is not continuous on $[0, 2]$.

Example. Another potential problem is inability to find antiderivative, such as for

$$\int_0^1 e^{x^2}\,dx$$

In this case, we have to use an estimation method. It turns out that this integral equals, to 30 decimal places:

1.462651745907181608804048586857

Example. Compute:

(a) $\dfrac{d}{dx}\displaystyle\int_0^x \ln(1+t^2)\,dt$

(b) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \ln(1+t^2)\,dt$

(c) $\dfrac{d}{dx}\displaystyle\int_x^{10} \ln(1+t^2)\,dt$

(d) $\dfrac{d}{dx}\displaystyle\int_x^{x^2} \ln(1+t^2)\,dt$

(e) $\displaystyle\lim_{x\to 0} \frac{\int_0^x \ln(1+t^2)\,dt}{x^3}$

Example. Compute:

(a) $\dfrac{d}{dx}\displaystyle\int_0^x \ln(1+t^2)\,dt$

(b) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \ln(1+t^2)\,dt$

(c) $\dfrac{d}{dx}\displaystyle\int_x^{10} \ln(1+t^2)\,dt$

(d) $\dfrac{d}{dx}\displaystyle\int_x^{x^2} \ln(1+t^2)\,dt$

(e) $\displaystyle\lim_{x\to 0}\frac{\int_0^x \ln(1+t^2)\,dt}{x^3}$

Here, we practice the *first* FTC.

Example. Compute:

(a) $\frac{d}{dx}\int_0^x \ln(1+t^2)\,dt = \ln(1+x^2)$

(b) $\frac{d}{dx}\int_0^{x^2} \ln(1+t^2)\,dt = \ln(1+x^4)\cdot 2x$

(c) $\frac{d}{dx}\int_x^{10} \ln(1+t^2)\,dt = -\ln(1+x^2)$

(d) $\frac{d}{dx}\int_x^{x^2} \ln(1+t^2)\,dt = \ln(1+x^4)\cdot 2x - \ln(1+x^2)$

(e) $\lim_{x\to 0} \frac{\int_0^x \ln(1+t^2)\,dt}{x^3} = \lim_{x\to 0}\frac{\ln(1+x^2)}{3x^2} = \lim_{x\to 0}\frac{\frac{1}{1+x^2}\cdot 2x}{6x} = \frac{1}{3}$

Example. Compute

$$\int \cos(x^2) \cdot 2x \, dx$$

We have this situation:

$$\int f(g(x)) \cdot g'(x) \, dx$$

We can handle such integral by the substitution method, which is an application of the chain rule.

5.5. Substitution in indefinite integrals

Theorem (The substitution rule)

Assume that $F(x)$ is the antiderivative of $f(x)$. Then $F(g(x))$ is the antiderivative of $f(g(x)) \cdot g'(x)$.

Proof.

By the chain rule

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x).$$

□

Commonly, we use another variable, like u, to “substitute” for $g(x)$.

Example. Compute

$$\int \cos(x^2) \cdot 2x\, dx$$

If $u = g(x) = x^2$, the $du = g'(x)\, dx = 2x\, dx$. So we need to compute exactly the antiderivative of
$\cos(g(x)) \cdot g'(x)\, dx = \cos(u)\, du$.

The answer, by the theorem (applied to $f = \cos$ and $g(x) = x^2$), is

$$\sin u + C = \sin(g(x)) + C = \sin(x^2) + C.$$

Example. Compute

$$\int \frac{7x^2}{x^3 + 5}\, dx$$

Example. Compute

$$\int \frac{7x^2}{x^3+5}\,dx$$

$$\int \frac{7x^2}{x^3+5}\,dx = \frac{7}{3}\int \frac{3x^2}{x^3+5}\,dx$$

Now $u = x^3 + 5$, $du = 3x^2\,dx$, gives

$$\frac{7}{3}\int \frac{3x^2}{x^3+5}\,dx = \frac{7}{3}\int \frac{1}{u}\,du = \frac{7}{3}\ln|u| + C = \frac{7}{3}\ln|x^3+5| + C$$

Example. Compute

$$\int (3x+5)^{10}\,dx$$

Example. Compute

$$\int (3x+5)^{10}\,dx$$

Now, $u = 3x + 5$, $du = 3\,dx$, $dx = \frac{1}{3}\,du$, gives

$$\frac{1}{3}\int u^{10}\,du = \frac{1}{33}u^{11} + C = \frac{1}{33}(3x+5)^{11} + C$$

Example. Compute

$$\int \frac{x^3}{\sqrt{5x^2+7}}\, dx$$

Example. Compute

$$\int \frac{x^3}{\sqrt{5x^2+7}}\,dx$$

$$\int \frac{x^3}{\sqrt{5x^2+7}}\,dx = \int \frac{x^2}{\sqrt{5x^2+7}} \cdot x\,dx$$

Example. Compute

$$\int \frac{x^3}{\sqrt{5x^2+7}}\,dx$$

$$\int \frac{x^3}{\sqrt{5x^2+7}}\,dx = \int \frac{x^2}{\sqrt{5x^2+7}} \cdot x\,dx$$

Now $u = 5x^2 + 7$, $du = 10x\,dx$, $\frac{1}{10}du = x\,dx$, $x^2 = \frac{1}{5}(u-7)$, gives

$$\frac{1}{50}\int \frac{u-7}{\sqrt{u}}\,du$$

$$
\begin{aligned}
&\frac{1}{50}\int \frac{u-7}{\sqrt{u}}\,du \\
&= \frac{1}{50}\int (u^{1/2} - 7u^{-1/2})\,du \\
&= \frac{1}{50}\left(\frac{1}{3/2}u^{3/2} - 7\frac{1}{1/2}u^{1/2}\right) + C \\
&= \frac{1}{50}\left(\frac{2}{3}u^{3/2} - 14u^{1/2}\right) + C \\
&= \frac{1}{50}\left(\frac{2}{3}(5x^2+7)^{3/2} - 14(5x^2+7)^{1/2}\right) + C
\end{aligned}
$$

Example. Compute

$$\int \frac{x^2}{(2x+1)^3}\,dx$$

Example. Compute

$$\int \frac{x^2}{(2x+1)^3}\,dx$$

Now, $u = 2x+1$, $du = 2\,dx$, $dx = \frac{1}{2}\,du$, $x = \frac{1}{2}(u-1)$, gives

$$\begin{aligned}
\frac{1}{8}\int \frac{(u-1)^2}{u^3}\,du &= \frac{1}{8}\int \frac{u^2-2u+1}{u^3}\,du \\
&= \frac{1}{8}\int \left(u^{-1} - 2u^{-2} + u^{-3}\right)\,du \\
&= \frac{1}{8}\left(\ln|u| + 2u^{-1} - \frac{1}{2}u^{-2}\right) + C
\end{aligned}$$

and substitute back $u = 2x+1$ for the final answer.