Week 3 Lectures

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Example. Compute the area under the graph of $y = x\sqrt{x+1}$ on $[0, 1]$.

This is a positive function, so by definition, we need to compute

$$\int_0^1 x\sqrt{x+1}\,dx$$

5.6. Substitution in definite integral and area

To compute the indefinite integral

$$\int x\sqrt{x+1}\,dx,$$

the substitution is $u = x + 1$, $du = dx$, $x = u - 1$, so that

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (u-1)u^{1/2}\,du \\ &= \int (u^{3/2} - u^{1/2})\,du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\end{aligned}$$

We can now now substitute back: $u = x + 1$, and plug in $x = 1$ and $x = 0$, and subtract:

$$\begin{aligned}\int_0^1 x\sqrt{x+1}\,dx &= \left(\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2}\right)\Big|_{x=0}^{x=1} \\ &= \left(\frac{2}{5}2^{5/2} - \frac{2}{3}2^{3/2}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{4}{15}(\sqrt{2}+1)\end{aligned}$$

Another possibility is to change the bounds for x to bounds for u immediately after substitution: since $u = x + 1$, $\begin{array}{c|c} x & u \\ \hline 0 & 1 \\ 1 & 2 \end{array}$ which leads to exact same numbers, but with fewer steps:

$$\begin{aligned}
\int_0^1 x\sqrt{x+1}\,dx &= \int_1^2 (u-1)u^{1/2}\,du \\
&= \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_{u=1}^{u=2} \\
&= \left(\frac{2}{5}2^{5/2} - \frac{2}{3}2^{3/2}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)
\end{aligned}$$

This is typically the better option.

Example. Compute the area under $y = \tan x$ on $[0, \pi/4]$.

Again, $y = \tan x$ is nonnegative on $[0, \pi/4]$, so we compute

$$\int_0^{\pi/4} \tan x \, dx = \int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx$$

We use $u = \cos x$, $du = -\sin x \, dx$,

x	u
0	1
$\pi/4$	$\sqrt{2}/2$

so we get

$$\int_1^{\sqrt{2}/2} -\frac{1}{u} \, du = \int_{\sqrt{2}/2}^{1} \frac{1}{u} \, du$$

$$= \ln u \Big|_{u=\sqrt{2}/2}^{u=1} = -\ln \frac{\sqrt{2}}{2} = \frac{1}{2} \ln 2$$

5.6. Substitution in definite integral and area

Suppose you have an *odd* (continuous) function $y = f(x)$ on a symmetric interval $[-a, a]$. Then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Proof.

Consider the integral on $[-a, 0]$ and introduce $u = -x$,

$du = -dx$,

x	u
-a	a
0	0

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-u)\,du = -\int_{0}^{a} f(u)\,du = -\int_{0}^{a} f(x)\,dx$$

and so $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$ □

Similarly, if you have an *even* (continuous) function $y = f(x)$ on a symmetric interval $[-a, a]$. Then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Example. Compute

$$\int_{-1}^{1} e^{-x^4} \sin x\,dx$$

Similarly, if you have an *even* (continuous) function $y = f(x)$ on a symmetric interval $[-a, a]$. Then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Example. Compute

$$\int_{-1}^{1} e^{-x^4} \sin x\,dx$$

The answer is 0, because the function $y = e^{-x^4} \sin x$ is odd.

5.6. Substitution in definite integral and area

If we want to compute the area between two curves, the upper curve $y = f(x)$ and the lower curve $y = g(x)$ on $[a, b]$, we can cut the region into vertical ribbons: partition $[a, b]$ into intervals $[x_{k-1}, x_k]$ of width Δx_k, choose the left endpoint x_{k-1} of $[x_{k-1}, x_k]$, and then the area is approximated by

$$\sum_{k=1}^{n} (f(x_{k-1}) - g(x_{k-1}))\Delta x_k$$

and so the area equals

$$\int_a^b (f(x) - g(x))\, dx.$$

Example. Compute the area of the *bounded* region, bounded by the graphs $y = x$ and $y = x^2$.

These two graphs intersect at $x = 0$ and $x = 1$, and the upper curve is $y = x$. So we get

$$\int_0^1 (x - x^2)\, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Example. Compute the area of the *bounded* region, bounded by the curves $y = x^4 + 2 - x^2$ and $y = x^4 - x$.

Intersections: $2 - x^2 = -x$, $x^2 - x - 2 = 0$, $(x-2)(x+1) = 0$, $x = -1, 2$.
On $[-1, 2]$, $x^4 + 2 - x^2$ is the larger of the two functions, as we can check at the single x, say $x = 0$. The answer is

$$\int_{-1}^{2} ((x^4 + 2 - x^2) - (x^4 - x))\,dx$$
$$= \int_{-1}^{2} (-x^2 + x + 2)\,dx$$
$$= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Big|_{x=-1}^{x=2} = \ldots = \frac{9}{2}$$

We can also cut a region between curves $x = g(y)$ (left curve) and $x = f(y)$ (right curve), between $y = c$ and $y = d$ into horizontal strips. The area of one strip at y is approximated by $(f(y) - g(y))\Delta y$, and so the area equals

$$\int_c^d (f(y) - g(y))\, dy$$

Example. Compute the area bounded by $y = \sqrt{x}$, $y = 2\sqrt{x-12}$ and $y = 0$.

Intersection: $\sqrt{x} = 2\sqrt{x-12}$, $x = 4(x-12)$, $3x = 48$, $x = 16$.

5.6. Substitution in definite integral and area

We can write the area as

$$\int_0^{12} (\sqrt{x} - 0)\, dx + \int_{12}^{16} (\sqrt{x} - 2\sqrt{x-12})\, dx$$

Or, we can use that:

- the left curve $y = \sqrt{x}$ is $x = y^2$; and
- the right curve $y = 2\sqrt{x-12}$ is $x = y^2/4 + 12$.

This gives the area as

$$\int_0^4 \left(\frac{y^2}{4} + 12 - y^2 \right) = \int_0^4 \left(12 - \frac{3y^2}{4} \right)\, dy = \ldots = 32$$