Week 4 Lectures

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Product formula:

$$(uv)' = u'v + uv',$$

or in differential form

$$d(uv) = u\,dv + v\,du,$$

then rearranged

$$u\,dv = d(uv) - v\,du,$$

and finally integrated

$$\int u\,dv = uv - \int v\,du.$$

8.2. Integration by parts

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

This formula is useful when you integrate a product of two functions:

- the first (u), simplifies by differentiation; and
- the second (dv) does not get overly complicated by integration.

Example. Compute

$$\int xe^{2x}\,dx$$

$$\int xe^{2x}\,dx = (*)$$

Take

$$u = x, du = dx$$
$$dv = e^{2x}\,dx, v = \frac{1}{2}e^{2x}$$

so that

$$(*) = x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$
$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

Example. Compute

$$\int x^2 \ln x \, dx$$

Example. Compute

$$\int x^2 \ln x \, dx = (*)$$

Take

$$u = \ln x, du = \frac{1}{x}\, dx$$
$$dv = x^2\, dx, v = \frac{1}{3}x^3$$

so that

$$\begin{aligned}(*) &= \frac{1}{3}x^3 \cdot \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\, dx \\ &= \frac{1}{3}x^3 \cdot \ln x - \frac{1}{3}\int x^2\, dx \\ &= \frac{1}{3}x^3 \cdot \ln x - \frac{1}{9}x^3 + C\end{aligned}$$

Example. Compute

$$\int_0^1 \arctan x \, dx$$

Example. Compute

$$\int_0^1 \arctan x \, dx = (*)$$

Take

$$u = \arctan x, du = \frac{1}{1+x^2} dx$$
$$dv = dx, v = x$$

so that

$$(*) = x \arctan x \Big|_{x=0}^{x=1} - \int_0^1 \frac{x}{1+x^2} dx$$
$$= \frac{\pi}{4} - 0 - \int_0^1 \frac{x}{1+x^2} dx$$

$$(*) = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx$$

The remaining integral can be done by using substitution (it is preferable to not use the letter u again in the same problem):

$t = 1 + x^2$, $dt = 2x\,dx$, $x\,dx = \frac{1}{2}dt$,

x	t
0	1
1	2

.

$$\begin{aligned}(*) &= \frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{1}{t}\,dt \\ &= \frac{\pi}{4} - \frac{1}{2}\ln t\Big|_{t=1}^{t=2} \\ &= \frac{\pi}{4} - \frac{1}{2}\ln 2\end{aligned}$$

Example. Compute

$$\int_0^{\pi} x^2 \sin x \, dx$$

Example. Compute

$$\int_0^{\pi} x^2 \sin x \, dx = (*)$$

Take

$$u = x^2, du = 2x \, dx$$
$$dv = \sin x \, dx, v = -\cos x$$

so that

$$(*) = -x^2 \cos x \Big|_{x=0}^{x=\pi} + \int_0^{\pi} 2x \cos x \, dx$$
$$= \pi^2 + \int_0^{\pi} 2x \cos x \, dx$$

$$(*) = \pi^2 + \int_0^{\pi} 2x \cos x \, dx$$

The remaining integral can be done again by using integration by parts:

$$u = 2x, du = 2\, dx$$
$$dv = \cos x \, dx, v = \sin x$$

so that

$$\begin{aligned}(*) &= \pi^2 + 2x \sin x \Big|_{x=0}^{x=\pi} - 2 \int_0^{\pi} \sin x \, dx \\ &= \pi^2 + 0 + 2 \cos x \Big|_{x=0}^{x=\pi} \\ &= \pi^2 - 4\end{aligned}$$

Example. This is a famous tricky example. Compute

$$\int e^x \sin x \, dx$$

Example. Compute

$$\int e^x \sin x \, dx = (*)$$

$$u = \sin x, du = \cos x \, dx$$
$$dv = e^x \, dx, v = e^x$$

$$(*) = e^x \sin x - \int e^x \cos x \, dx$$
$$u = \cos x, du = -\sin x \, dx$$
$$dv = e^x \, dx, v = e^x$$

$$(*) = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

We have an equation for $\int e^x \sin x \, dx$, which we can solve:

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

Example. Compute

$$\int_0^{\pi/3} \sin x \cdot \sin 5x \, dx$$

For examples like this, we use the formulas:

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$
$$\sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$$
$$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$$

So,

$$\sin x \cdot \sin 5x = \frac{1}{2}(\cos 4x - \cos 6x)$$

Example. Compute

$$\begin{aligned}
&\int_0^{\pi/3} \sin x \cdot \sin 5x \, dx \\
&= \frac{1}{2} \int_0^{\pi/3} (\cos 4x - \cos 6x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{4} \sin 4x - \frac{1}{6} \sin 6x \right) \Big|_{x=0}^{x=\pi/3} \\
&= \frac{1}{2} \left(\frac{1}{4} \sin \frac{4\pi}{3} - \frac{1}{6} \sin 2\pi \right) \\
&= -\frac{1}{8} \sin \frac{\pi}{3} = -\frac{\sqrt{3}}{16}
\end{aligned}$$

Example. Compute

$$\int \sin^5 x \cdot \cos^2 x \, dx$$

In integrals $\int \sin^m x \cdot \cos^n x \, dx$, when one of the two powers m and n is an odd positive integer, substitution works.
Write $\sin^5 x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x$.
The integral becomes

$$\int (1 - \cos^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$$

The integral becomes

$$\int (1 - \cos^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$$

Now $u = \cos x$, $du = -\sin x \, dx$, results in

$$\begin{aligned} -\int (1 - u^2)^2 u^2 \, du &= -\int (1 - 2u^2 + u^4) u^2 \, du \\ &= -\int (u^2 - 2u^4 + u^6) \, du \\ &= -\frac{1}{3} u^3 + \frac{2}{5} u^5 - \frac{1}{7} u^7 + C \end{aligned}$$

and substitute back $u = \cos x$.

Example. Compute

$$\int \sin^4 x \, dx$$

In integrals $\int \sin^m x \cdot \cos^n x \, dx$, when the two powers m and n are even nonnegative integers, use the double-angle formulas:

$$\sin^2 A = \frac{1}{2}(1 - \cos(2A)),$$
$$\cos^2 A = \frac{1}{2}(1 + \cos(2A)).$$

The integral becomes:

$$\int \frac{1}{4}(1 - \cos 2x)^2 \, dx$$

$$
\begin{aligned}
\int \frac{1}{4}(1-\cos 2x)^2\,dx &= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx \\
&= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right)dx \\
&= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)\,dx \\
&= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C
\end{aligned}
$$

Example. Which substitutions would work in
(a) $\int \tan^5 x \, dx$, (b) $\int \tan^4 x \, dx$.

Example. Which substitutions would work in
(a) $\int \tan^5 x\, dx$.

Write

$$\tan^5 x = \frac{\sin^5 x}{\cos^5 x} = \frac{(1-\cos^2 x)^2}{\cos^5 x} \cdot \sin x$$

The substitution $u = \cos x$ works and leads to:

$$-\int \frac{(1-u^2)^2}{u^5}\, du$$

Example. Which substitutions would work in

(b) $\displaystyle\int \tan^4 x \, dx$.

Write

$$\tan^4 x = \tan^2 x \cdot \sin^2 x \cdot \frac{1}{\cos^2 x} = \tan^2 x \frac{\tan^2 x}{1 + \tan^2 x} \cdot \frac{1}{\cos^2 x}$$

The substitution $u = \tan x$ works, but leads to integral

$$\int \frac{u^4}{1 + u^2} \, du$$

which can be solved by long division:

$$\int \frac{u^4}{1 + u^2} \, du = \int \left(u^2 - 1 + \frac{1}{u^2 + 1} \right) \, du$$

We will do such integrals in detail later.

Example. Compute

$$\int_{-\pi}^{\pi} \sqrt{1 - \cos x}\, dx$$

We observe

$$1 - \cos x = 2 \sin^2 \frac{x}{2}$$

so that the integral equals

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sqrt{2} \left| \sin \frac{x}{2} \right| \, dx \\
&= 2\sqrt{2} \int_{0}^{\pi} \sin \frac{x}{2}\, dx \\
&= -4\sqrt{2} \cos \frac{x}{2} \bigg|_{x=0}^{x=\pi} = 4\sqrt{2}
\end{aligned}$$

Integrals of sec and csc:

$$\int \frac{1}{\cos x}\,dx = \ln\left|\frac{1}{\cos x} + \tan x\right| + C$$

$$\int \frac{1}{\sin x}\,dx = -\ln\left|\frac{1}{\sin x} + \cot x\right| + C$$

Proof of:

$$\int \frac{1}{\cos x}\,dx = \ln\left|\frac{1}{\cos x} + \tan x\right| + C$$

Write

$$\int \frac{1}{\cos x}\,dx = \int \frac{\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}}{\frac{1}{\cos x} + \tan x}\,dx$$

If

$$u = \frac{1}{\cos x} + \tan x, \quad du = \left(\frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x}\right)dx$$

so we get

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

We can get rid of square roots $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$, by using substitutions below.

$\sqrt{\ }$	int. for x	subst. θ	int. for θ	$\sqrt{\ }$ becomes	dx
$\sqrt{a^2 - x^2}$	$(-a, a)$	$x = a \sin\theta$	$(-\pi/2, \pi/2)$	$\sqrt{a^2 - x^2} = a\cos\theta$	$dx = a\cos\theta\, d\theta$
$\sqrt{a^2 + x^2}$	$(-\infty, \infty)$	$x = a\tan\theta$	$(-\pi/2, \pi/2)$	$\sqrt{a^2 + x^2} = a\sec\theta$	$dx = a\sec^2\theta\, d\theta$
$\sqrt{x^2 - a^2}$	(a, ∞)	$x = a\sec\theta$	$(0, \pi/2)$	$\sqrt{x^2 - a^2} = a\tan\theta$	$dx = a\sec\theta\tan\theta\, d\theta$

Example.

$\sqrt{\ }$	int. for x	subst. θ	int. for θ	$\sqrt{\ }$ becomes	dx
$\sqrt{a^2 - x^2}$	$(-a, a)$	$x = a\sin\theta$	$(-\pi/2, \pi/2)$	$\sqrt{a^2 - x^2} = a\cos\theta$	$dx = a\cos\theta\, d\theta$

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx &= \int a^2 \cos^2\theta\, d\theta = \frac{a^2}{2}\int (1 + \cos 2\theta)\, d\theta \\ &= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\ &= \frac{a^2}{2}\theta + \frac{a^2}{4} 2\sin\theta\sqrt{1 - \sin^2\theta} + C \\ &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\frac{x}{a}\sqrt{1 - \left(\frac{x}{a}\right)^2} + C \\ &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C\end{aligned}$$

Example.

$\sqrt{\ }$	int. for x	subst. θ	int. for θ	$\sqrt{\ }$ becomes	dx
$\sqrt{a^2+x^2}$	$(-\infty,\infty)$	$x=a\tan\theta$	$(-\pi/2,\pi/2)$	$\sqrt{a^2+x^2}=a\sec\theta$	$dx=a\sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{\cos\theta}{2}\cdot\frac{2}{\cos^2\theta}\,d\theta = \int\frac{1}{\cos\theta}\,d\theta$$

$$= \ln|\sec\theta+\tan\theta| + C$$

$$= \ln\left|\sqrt{1+\tan^2\theta}+\tan\theta\right| + C$$

$$= \ln\left|\sqrt{1+\left(\frac{x}{2}\right)^2}+\frac{x}{2}\right| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}+x}{2}\right| + C$$

$$= \ln\left(\sqrt{4+x^2}+x\right) + C$$