Week 5 Lectures

Janko Gravner

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We will discuss how to compute integrals of the form

$$\int \frac{F(x)}{G(x)}\,dx$$

where $F(x)$ and $G(x)$ are polynomials in x. These are integrals of *rational functions*. Such integrals can be reduced to three types of integrals that we address first.

Type 1. Denominator is linear to some power: $\displaystyle\int \frac{p(x)}{(ax+b)^n}\,dx$

Here $a \neq 0$.
We *always* substitute $u = ax + b$.

Example. Compute $\int \frac{x}{(5x+3)^4}\,dx$

Example. Compute $\int \frac{x}{(5x+3)^4}\,dx = (*)$

Let

$$u = 5x+3, du = 5\,dx, dx = \frac{1}{5}\,du, x = \frac{1}{5}(u-3).$$

Then

$$\begin{aligned}(*) = \frac{1}{25}\int \frac{(u-3)}{u^4}\,du &= \frac{1}{25}\int \frac{du}{u^3} - \frac{3}{25}\int \frac{du}{u^4} \\ &= \frac{1}{25}\frac{u^{-2}}{(-2)} - \frac{3}{25}\frac{u^{-3}}{(-3)} + C \\ &= -\frac{1}{50}\frac{1}{(5x+3)^2} + \frac{1}{25}\frac{1}{(5x+3)^3} + C\end{aligned}$$

Type 2. Irreducible quadratic denominator: $\int \frac{mx+n}{ax^2+bx+c}\,dx$
Here the discriminant is negative, that is,
$b^2-4ac<0$. The denominator cannot be factored.
We *always* complete the square in the denominator and substitute

(linear expression inside the square)=
(square root of the constant summand)$\cdot u$

Example. Compute $\int \frac{1}{x^2+1}\,dx$

No square completion or substitution necessary; the answer is $\arctan x + C$.

Example. Compute $\displaystyle\int \frac{x}{3x^2+2x+1}\,dx = (*)$

Complete the square:

$$\begin{aligned} 3x^2+2x+1 &= 3\left(x^2+\frac{2}{3}x+\frac{1}{3}\right) \\ &= 3\left(\left(x+\frac{1}{3}\right)^2-\frac{1}{9}+\frac{1}{3}\right) \\ &= 3\left(\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right) \end{aligned}$$

Substitute u so that $(x+\frac{1}{3})^2 = \frac{2}{9}u^2$.

$$x+\frac{1}{3} = \frac{\sqrt{2}}{3}u, dx = \frac{\sqrt{2}}{3}\,du, x = \frac{\sqrt{2}}{3}u - \frac{1}{3}$$

This results in

$$\begin{aligned}
(*) &= \frac{\sqrt{2}}{3} \int \frac{\frac{\sqrt{2}}{3}u - \frac{1}{3}}{3 \cdot \frac{2}{9} \cdot (u^2+1)}\, du \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{3} \int \frac{u}{u^2+1}\, du - \frac{\sqrt{2}}{6} \int \frac{1}{u^2+1}\, du \\
&= \frac{1}{3} \cdot \frac{1}{2} \ln(u^2+1) - \frac{\sqrt{2}}{6} \arctan u + C \\
&= \frac{1}{6} \ln(\left(\frac{3}{\sqrt{2}}\left(x+\frac{1}{3}\right)\right)^2 + 1) - \frac{\sqrt{2}}{6} \arctan\left(\frac{3}{\sqrt{2}}\left(x+\frac{1}{3}\right)\right) + C
\end{aligned}$$

Type 3. Irreducible quadratic expression to some power denominator: $\int \frac{mx+n}{(ax^2+bx+c)^k}\,dx$
Here $a \neq 0$ and the discriminant is negative, that is, $b^2 - 4ac < 0$. The denominator cannot be further factored.
As in the last example, we factor out the constant in front of x^2, then complete the square in the denominator and substitute,

(linear expression inside the square)=
(square root of the constant summand)$\cdot u$

Integration by parts can be used to reduce the power k.

Example. Compute $\int \frac{x+1}{(x^2+1)^2}\,dx$

No square completion or substitution necessary. We separate into two integrals: $\int \frac{x}{(x^2+1)^2}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$

The first one is an easy substitution $u = x^2 + 1$, $du = 2x\,dx$,

$$\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}u^{-1} + C = -\frac{1}{2}\,\frac{1}{x^2+1} + C$$

so we concentrate on the second.

$$\begin{aligned}\int \frac{1}{(x^2+1)^2}\,dx &= \int \frac{x^2+1-x^2}{(x^2+1)^2}\,dx \\ &= \int \frac{1}{x^2+1}\,dx - \int \frac{x^2}{(x^2+1)^2}\,dx \\ &= \arctan x - \int x \cdot \frac{x}{(x^2+1)^2}\,dx = (*)\end{aligned}$$

By parts in the remaining integral $\int x \cdot \frac{x}{(x^2+1)^2}\,dx$

$$u = x, du = dx$$
$$dv = \frac{x}{(x^2+1)^2}\,dx, v = -\frac{1}{2}\frac{1}{x^2+1}$$

$$(*) = \arctan x + \frac{1}{2}\frac{x}{x^2+1} - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$
$$= \frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{x^2+1} + C$$

Final answer:

$$\int \frac{x+1}{(x^2+1)^2}\,dx = -\frac{1}{2}\frac{1}{x^2+1} + \frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{x^2+1} + C$$

We are computing integrals of the form

$$\int \frac{F(x)}{G(x)}\,dx$$

where $F(x)$ and $G(x)$ are polynomials in x. We will now describe the *partial fraction expansion* of $\frac{F(x)}{G(x)}$, which equals to the sum of expressions described below.

Step 0. If the degree of the numerator $F(x)$ is larger then or equal to the degree of the denominator $G(x)$, use long division to reduce the degree of the numerator. We will, from now on, assume that the degree of $F(x)$ is strictly smaller than the degree of $G(x)$.

8.5. Partial fractions

Step 1. Factor the denominator $G(x)$ into factors of the form $(x-r)^m$ and irreducible factors $(x^2+px+q)^n$. The decomposition of $\frac{F(x)}{G(x)}$ is the sum of contributions from each of these factors.

Step 2. Each factor $(x-r)^m$ contributes this:

$$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$$

Step 3. Each factor $(x^2+px+q)^n$ contributes this:

$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}$$

Step 4. Compute constants A, B, C.

In order to compute the integral:

Step 5. Compute the resulting integrals of Type 1–3.

Example. What is the PFE of

$$\frac{1}{(1-x^4)(1-x)^2}$$

The denominator is not yet factored, so we do this first

$$\begin{aligned}(1-x^4)(1-x)^2 &= (1-x^2)(1+x^2)(1-x)^2 \\ &= (1-x)(1+x)(1+x^2)(1-x)^2 \\ &= (1+x)(1-x)^3(1+x^2)\end{aligned}$$

and then write out

$$\frac{1}{(1+x)\cdot(1-x)^3\cdot(1+x^2)}$$
$$= \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} + \frac{D}{(1-x)^3} + \frac{Ex+F}{1+x^2}.$$

Example. Compute $\int \frac{x^3+2}{x^2-x}\,dx = (*)$

Long division:

$$(*) = \int \left(x + 1 + \frac{x+2}{x^2-x} \right)\,dx$$

PFE:

$$\frac{x+2}{x^2-x} = \frac{x+2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$*x$, $x \to 0$: $A = -2$;

$*(x-1)$, $x \to 1$: $B = 3$.

$$\frac{x+2}{x^2-x} = -\frac{2}{x} + \frac{3}{x-1}$$

$$
\begin{aligned}
(*) &= \int \left(x + 1 - \frac{2}{x} + \frac{3}{x-1} \right) \, dx \\
&= \frac{1}{2}x^2 + x - 2\ln|x| + 3\ln|x-1| + C
\end{aligned}
$$

Example. Compute $\int \frac{1}{x^3 + 4x^2 - 5x}\, dx = (*)$

Factor:

$$(*) = \int \frac{1}{x(x+5)(x-1)}\, dx$$

PFE:

$$\frac{1}{x(x+5)(x-1)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-1}$$

$*x$, $x \to 0$: $A = -1/5$;

$*(x+5)$, $x \to -5$: $B = 1/30$;

$*(x-1)$, $x \to 1$: $C = 1/6$.

$$\frac{1}{x(x+5)(x-1)} = \frac{-1/5}{x} + \frac{1/30}{x+5} + \frac{1/6}{x-1}$$

$$
\begin{aligned}
(*) &= \int \left(\frac{-1/5}{x} + \frac{1/30}{x+5} + \frac{1/6}{x-1} \right) \, dx \\
&= -\frac{1}{5} \ln|x| + \frac{1}{30} \ln|x+5| + \frac{1}{6} \ln|x-1| + C
\end{aligned}
$$

Example. Write the PFE for $\int \frac{x^2+1}{x^3+x^2+2x}\,dx = (*)$

Factor:

$$(*) = \int \frac{x^2+1}{x(x^2+x+2)}\,dx$$

PFE:

$$\frac{x^2+1}{x(x^2+x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+2}$$

$*x$, $x \to 0$: $A = 1/2$;

$*x$, $x \to \infty$: $A + B = 1$, $B = 1/2$;

$x = 1$: $1/2 = A + B/4 + C/4$, $C = 2 - 4A - B = -1/2$

$$\frac{x^2+1}{x(x^2+x+2)} = \frac{1}{2}\frac{1}{x} + \frac{1}{2}\frac{x-1}{x^2+x+2}$$

$$(*) = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{x-1}{x^2+x+2}\,dx$$

The first integral is $\ln|x|$ and the second is handled by $x^2+x+2 = (x+\frac{1}{2})^2 + \frac{7}{4}$ and $x + \frac{1}{2} = \frac{\sqrt{7}}{2}u$.

Example. Compute $\int \frac{x}{(x^2+1)(x+1)^2}\,dx = (*)$

PFE:

$$\frac{x}{(x^2+1)(x+1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$*(x+1)^2$, $x \to -1$: $D = -1/2$;

$*(x^2+1)$, $x \to i$: (note that $(i+1)^2 = 2i$) $1/2 = Ai + B$, $A = 0$, $B = 1/2$;

$*x, x \to \infty$: $0 = A + C$, $C = 0$

$$\frac{x}{(x^2+1)(x+1)^2} = \frac{1/2}{x^2+1} - \frac{1/2}{(x+1)^2}$$

$$(*) = \frac{1}{2}\int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{(x+1)^2}\,dx$$
$$= \frac{1}{2}\arctan x + \frac{1}{2}\frac{1}{x+1} + C$$

Example. Compute $\int \frac{1}{x+\sqrt{x+1}+3}\,dx = (*)$

$$x+1=u^2, \quad dx = 2u\,du, \quad x = u^2-1$$

$$(*) = \int \frac{2u}{u^2+u+2}\,du = \cdots$$

(Complete the square.)

Example. Compute $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx = (*)$

$$x = u^6, \quad dx = 6u^5\, du$$

$$(*) = \int \frac{6u^5}{u^3 + u^2}\, du = \int \frac{6u^3}{u+1}\, du \cdots$$

(Long division, or substitute $z = u + 1$.)