Week 6 Lectures

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6.1. Volumes using cross-sections

Assume that we cut a body with a plane perpendicular to some axis, which we call the x-axis. If that plane is at x, call the area of that cross-section $A(x)$. Assume that the body extends from $x = a$ to $x = b$.

A prism with base of area $A(x)$ and height Δx has volume $A(x)\Delta x$.

Now imagine that the body is cut into thin slices with planes at x_i, of width Δx_i. Then the volume of the body is approximated by $\sum_{i=1}^{n} A(x_i)\Delta x_i$ and therefore equals $\int_a^b A(x)\,dx$.

6.1. Volumes using cross-sections

Suppose you revolve a region between two graphs $y = R(x)$ (upper graph) and $y = r(x)$ (lower graph), on the interval $[a, b]$, around the x-axis. How do we compute the volume of this region?

The resulting cross-section at x (by a plane perpendicular to the x-axis) is the region between two circles, the outer circle with radius $R(x)$ and the inner circle with radius $r(x)$. Its area is $\pi R(x)^2 - \pi r(x)^2$, and so

$$\begin{aligned} \text{Volume} &= \int_a^b \pi (R(x)^2 - r(x)^2)\, dx \\ &= \int_a^b \pi ((\text{outer radius})^2 - (\text{inner radius})^2)\, dx \end{aligned}$$

We are dividing the region into thin “washers,” so this is called the *washer* method.

Example. Rotate the region below the graph of $y = x^2$ on $[0, 1]$ around the x-axis. Compute the volume.

$$\text{Volume} = \pi \int_0^1 x^4 \, dx = \frac{\pi}{5}$$

Example. Rotate the region bounded by the graphs of $y = x^2$ and $y = 3x - 2$ around (a) the x-axis and (b) the line $y = 1$. Set up the integrals for the volume in each case.

Intersections: $x^2 - 3x + 2 = 0$, $(x - 2)(x - 1) = 0$, $x = 1, 2$. Intersections are $(1, 1)$ and $(2, 4)$, and $y = 3x - 2$ is the top function on $[1, 2]$. Both functions are increasing on $[1, 2]$, so no part of the region is below the line $y = 1$.

(a) $\pi \int_1^2 ((3x - 2)^2 - (x^2)^2)\, dx$
(Multiply out and integrate powers.)

(b) Now outer radius $= 3x - 2 - 1$, inner radius $= x^2 - 1$.
$\pi \int_1^2 ((3x - 3)^2 - (x^2 - 1)^2)\, dx$
(Again, multiply out and integrate powers.)

Example. Compute the volume of the ball with radius r.

We rotate the top part of the circle $x^2 + y^2 = r^2$ around the x-axis, that is, the region below the graph of $y = \sqrt{r^2 - x^2}$ on $[-r, r]$.

$$\begin{aligned} \text{Volume} = \pi \int_{-r}^{r} (r^2 - x^2)\, dx &= 2\pi \int_0^r (r^2 - x^2)\, dx \\ &= 2\pi \left(r^2 x - \frac{1}{3} x^3 \right) \Big|_0^r \\ &= \frac{4}{3} \pi r^3 \end{aligned}$$

6.2. Volumes using cylindrical shells

A region R bounded between $x = a$ and $x = b$ is rotated around the y-axis. Resulting volume?

Divide R into thin vertical strips; the strip at x has height $h(x)$ and width Δx. When rotated, such a strip generates a thin cylindrical shell. When cut vertically and unfolded, this shell is a thin rectangular plate with volume $2\pi x h(x)\Delta x$. So the volume is approximated by $\sum_{i=1}^{n} 2\pi x_i h(x_i)\Delta x_i$ and therefore equals

$$\begin{aligned} \text{Volume} &= \int_a^b 2\pi x\, h(x)\, dx \\ &= \int_a^b 2\pi (\text{radius of the shell})\,(\text{height of the shell})\, dx \end{aligned}$$

6.2. Volumes using cylindrical shells

We are dividing the region into thin shells, so this is called the *shell* method. For example, if we are rotating a region between the graphs of $f(x)$ (upper function) and $g(x)$ (lower function) around the y axis, the resulting volume is

$$\text{Volume} = \int_a^b 2\pi x\,(f(x) - g(x))\,dx$$

Example. Rotate the region below the graph of $y = x^2$ on $[0, 1]$ around the y-axis. Compute the volume.

$$\text{Volume} = 2\pi \int_0^1 x \cdot x^2\,dx = \frac{\pi}{2}$$

Example. Rotate the region bounded by the graphs of $y = x^2$ and $y = 3x - 2$ around (a) the y-axis and (b) the line $x = -1$. Set up the integrals for the volume in each case.

We know that intersections are $(1,1)$ and $(2,4)$, and $y = 3x - 2$ is the top function on $[1,2]$.

(a) $2\pi \int_1^2 x\,((3x-2) - x^2)\,dx$

(b) Now radius of the shell $= (x+1)$,
height of the shell $= ((3x-2) - x^2)$.
$2\pi \int_1^2 (x+1)\,((3x-2) - x^2)\,dx$

Example. Rotate the region bounded by $y = \sqrt{x}$,
$y = 2\sqrt{x - 12}$ and $y = 0$ around the x axis. Volume?

Intersection: $\sqrt{x} = 2\sqrt{x - 12}$, $x = 4(x - 12)$, $3x = 48$, $x = 16$.

We can write the volume by the washer method as

$$\pi \int_0^{12} x\,dx + \pi \int_{12}^{16} (x - 4(x - 12))\,dx$$

Or, we can use the shell method, using that:

- the left curve $y = \sqrt{x}$ is $x = y^2$; and
- the right curve $y = 2\sqrt{x - 12}$ is $x = y^2/4 + 12$.

This gives the area as

$$2\pi \int_0^4 y\left(\frac{y^2}{4} + 12 - y^2\right)\,dx = 2\pi \int_0^4 \left(12y - \frac{3y^3}{4}\right)\,dy$$
$$= \ldots = 96\pi$$

6.1–6.2. Volumes of solids of rotation, review

method	thin strips	integrate
washer	are *perpendicular* to axis of rotation make "washers" when rotated	π((outer radius)^2 $-$ (inner radius)^2)
shell	are *parallel* to axis of rotation make "shells" when rotated	2π(radius of shell)(height of shell)

6.1–6.2. Volumes of solids of rotation, review

Example. Consider the region bounded by $y = e^{2x}$, $y = e^x$ and $x = 1$. Set up the integral for the volume when this region is rotated around:

1. y-axis: $\quad 2\pi \int_0^1 x(e^{2x} - e^x)\,dx$
2. $x = -1$: $\quad 2\pi \int_0^1 (x+1)(e^{2x} - e^x)\,dx$
3. $x = 1$: $\quad 2\pi \int_0^1 (1-x)(e^{2x} - e^x)\,dx$
4. x-axis: $\quad \pi \int_0^1 (e^{4x} - e^{2x})\,dx$
5. $y = -1$: $\quad \pi \int_0^1 ((e^{2x}+1)^2 - (e^x+1)^2)\,dx$
6. $y = 9$: $\quad \pi \int_0^1 ((9-e^x)^2 - (9-e^{2x})^2)\,dx$