Week 7 Lectures

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MAT 21B
Fall 2023

6.3. Arc Length

We want to compute the length of the curve given by $y = f(x)$ for x in $[a, b]$.

The length of the curve Δs between x and $x + \Delta x$ is approximated by the length of the secant. So,

$$\Delta s^2 \approx \Delta x^2 + \Delta y^2, \quad \Delta s \approx \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

Further,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \approx f'(x)$$

so that

$$\Delta s \approx \sqrt{1 + f'(x)^2}\,\Delta x$$

6.3. Arc Length

Now imagine that the curve is cut into small pieces between x_i and $x_i + \Delta x_i$ of length Δs_i.

Then the total arc length is approximated by

$$\sum_{i=1}^{n} \Delta s_i \approx \sum_{i=1}^{n} \sqrt{1 + f'(x_i)^2}\, \Delta x_i$$

and therefore equals

$$\text{Arc Length} = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If we write the arc length from a to x as

$$s(x) = \int_a^x \sqrt{1 + f'(t)^2}\, dt,$$

then

$$\frac{ds}{dx} = \sqrt{1 + f'(x)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

which is sometimes suggestively written as

$$ds^2 = dx^2 + dy^2,$$

the differential form of the Pythagorean theorem.

Example. Compute the arc length of $y = 2x^{3/2}$ between $x = 0$ and $x = 1$.
We have $\frac{dy}{dx} = 3x^{1/2}$ and so

$$\begin{aligned} \text{Arc Length} &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1+9x}\, dx \\ &= \frac{1}{9}\,\frac{2}{3}\,(1+9x)^{3/2}\Big|_0^1 = \frac{2}{27}(10^{3/2} - 1) \end{aligned}$$

Arc length integrals are commonly impossible or at least very hard.

Example. Compute the arc length of the parabola $y = x^2$ between $x = 0$ and $x = 1$. Now $\frac{dy}{dx} = 2x$, so

$$\text{Arc Length} = \int_0^1 \sqrt{1 + 4x^2}\, dx = 2 \int_0^1 \sqrt{\frac{1}{4} + x^2}\, dx = (*)$$

We can use the formula

$$\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C$$

This can be obtained by a trigonometric substitution, similarly to a discussion problem.

This gives

$$(*) = 2\left[\frac{x}{2}\sqrt{\frac{1}{4}+x^2}+\frac{1}{8}\ln\left(x+\sqrt{\frac{1}{4}+x^2}\right)\right]\Bigg|_{x=0}^{x=1}$$

$$= 2\left[\frac{1}{2}\sqrt{\frac{5}{4}}+\frac{1}{8}\ln\left(1+\sqrt{\frac{5}{4}}\right)-\frac{1}{8}\ln\frac{1}{2}\right]$$

$$= \frac{\sqrt{5}}{2}+\frac{1}{4}\ln\left(2+\sqrt{5}\right) \approx 1.4789$$

Check: this needs to be between $\sqrt{2}$ (the diagonal of the square) and 2 (two sides of the square)!

We rotate the graph of $y = f(x)$ between $x = a$ and $x = b$ around the x axis. We want to compute the area of the resulting solid of revolution.

What area do we get when we rotate a line segment between x and $x + \Delta x$, of length Δs, around the x-axis? This is the area of a “frustrum” of a cone. If we cut and flatten the cone, we get the area between two circular segments. Let ℓ be the radius of the smaller circle so that $\ell + \Delta s$ is the radius of the larger one. Also denote by θ the angle of the opening of the segments. The smaller arc has length $2\pi f(x)$. The area of the frustrum then is

$$\begin{aligned}\frac{1}{2}(\ell + \Delta s)^2\theta - \frac{1}{2}\ell^2\theta &= \frac{1}{2}\theta(\ell^2 + 2\ell\Delta s + \Delta s^2 - \ell^2) \\ &\approx \theta\ell\Delta s \\ &= 2\pi f(x)\Delta s\end{aligned}$$

So, the surface area is approximated by

$$\sum_i 2\pi f(x_i)\Delta s_i \approx \sum_i 2\pi f(x_i)\sqrt{1+f'(x_i)^2}\,\Delta x_i$$

and so

$$\begin{aligned}\text{Surface Area} &= \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx \\ &= \int_a^b 2\pi y\,ds \\ &= \int_a^b 2\pi(\text{radius of rotation})\,ds\end{aligned}$$

Example. Rotate $y = \sqrt{x}$, x in $[1, 2]$, around the x-axis. Compute the resulting surface area.

We compute $f'(x) = \frac{1}{2\sqrt{x}}$.

$$\begin{aligned}
\text{Surface Area} &= \int_1^2 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx \\
&= \int_1^2 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx \\
&= \pi \int_1^2 \sqrt{4x+1}\, dx \\
&= \pi \cdot \frac{1}{4} \cdot \frac{2}{3}(4x+1)^{3/2}\Big|_{x=1}^{x=2} \\
&= \frac{\pi}{6} \cdot (9^{3/2} - 5^{3/2}) = \frac{\pi}{6} \cdot (27 - 5\sqrt{5})
\end{aligned}$$

Surface area integrals are typically even harder than arc length ones.

Example. Rotate $y = x^2$, x in $[0, 1]$, around the x-axis. Compute the resulting surface area.

We compute $f'(x) = 2x$.

$$\begin{aligned}
\text{Surface Area} &= \int_0^1 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx \\
&= \int_0^1 2\pi x^2\sqrt{1 + 4x^2}\,dx \\
&= 4\pi \int_0^1 x^2\sqrt{\frac{1}{4} + x^2}\,dx
\end{aligned}$$

We use the formula

$$\int x^2\sqrt{a^2+x^2}\,dx$$
$$=\frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2}-\frac{a^2}{8}\ln(x+\sqrt{a^2+x^2})+C$$

Surface Area

$$=4\pi\int_0^1 x^2\sqrt{\frac{1}{4}+x^2}\,dx$$
$$=4\pi\left[\frac{x}{8}\left(\frac{1}{4}+2x^2\right)\sqrt{\frac{1}{4}+x^2}-\frac{1}{128}\ln\left(x+\sqrt{\frac{1}{4}+x^2}\right)\right]\Bigg|_{x=0}^{x=1}$$
$$=\frac{9\pi\sqrt{5}}{16}-\frac{\pi}{32}\ln(2+\sqrt{5})\approx 3.8097$$

Example. Rotate $y = \sqrt{x}$, x in $[1, 2]$, around the line $y = -1$. Set up the integral for the resulting surface area.

We compute $f'(x) = \frac{1}{2\sqrt{x}}$. Recall:

$$\text{Surface Area} = \int_a^b 2\pi(\text{radius of rotation})\, ds$$

so in this case

$$\text{Surface Area} = \int_1^2 2\pi(\sqrt{x} + 1)\sqrt{1 + \frac{1}{4x}}\, dx$$

(It is possible to compute $\int \sqrt{1 + \frac{1}{4x}}\, dx$ by substitution $1 + \frac{1}{4x} = u^2$.)

Example. Rotate $y = e^x$, x in $[0, 1]$, (a) around the y-axis, (b) around the line $x = 2$. Set up the integral for the resulting surface area.

We have two options. If we integrate over x, the radius of rotation is simply x. As $\frac{dy}{dx} = e^x$, we get for (a)

$$\begin{aligned} \text{Surface Area} &= \int_0^1 2\pi(\text{radius of rotation})\, ds \\ &= \int_0^1 2\pi x\sqrt{1 + e^{2x}}\, dx \end{aligned}$$

and for (b)

$$\text{Surface Area} = \int_0^1 2\pi(2 - x)\sqrt{1 + e^{2x}}\, dx$$

The second option is to integrate over y, in which case we write the curve as $x = \ln y$. We have $\frac{dx}{dy} = \frac{1}{y}$ and $ds^2 = dx^2 + dy^2$ gives

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{1}{y^2}}\, dy$$

(a) So, integrating over the range of y, which is from 1 to e,

$$\begin{aligned} \text{Surface Area} &= \int_1^e 2\pi(\text{radius of rotation})\, ds \\ &= \int_1^e 2\pi \ln y \sqrt{1 + \frac{1}{y^2}}\, dy \end{aligned}$$

(b) The same approach now gives

$$\text{Surface Area} = \int_1^e 2\pi(2 - \ln y)\sqrt{1 + \frac{1}{y^2}}\, dy$$

Imagine a seesaw along the x-axis with the pivot point ("fulcrum") at $x = k$. Children with masses $m_1, m_2, \ldots, m_n$ sit at locations $x_1, x_2, \ldots, x_n$. The *moment* of such seesaw is the sum of all (mass)·(length of the lever) terms:

$$\sum_{i=1}^{n} m_i(x_i - k)$$

Multiplying also by g, the gravity acceleration, this gives the *torque*, or the rotational force, which measure the rotational tendency of the system: more precisely, it equals the derivative of the angular momentum.

In order for the system to be in the equilibrium, the torque must be 0, or equivalently

$$\sum_{i=1}^{n} m_i(x_i - k) = 0,$$

or

$$k = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i},$$

which we call the *center of mass*.

Assume that we have a thin plate shaped like a planar region R of constant density 1 (per unit area). Then the point $(\overline{x}, \overline{y})$ at which we can balance the plate is called the *centroid* of R.

Assume R extends between lines $x = a$ and $x = b$. Assume also that the length of the intersection between the vertical line at x and the region is $h(x)$.
We can compute the moment about the line $x = k$ as follows

$$\int_a^b (x - k)h(x)\,dx.$$

In particular, the moment about the y-axis ($x = 0$) is

$$M_y = \int_a^b xh(x)\,dx.$$

Therefore, $\overline{x}$ needs to satisfy

$$\int_a^b (x - \overline{x})h(x)\,dx = 0,$$

that is

$$\overline{x} = \frac{\int_a^b xh(x)\,dx}{\int_a^b h(x)\,dx} = \frac{M_y}{\text{Area}(R)}$$

Similarly, if the R extends between lines $y = c$ and $y = d$ and the length of the intersection between the horizontal line at y and the region is $w(y)$, then we define the the moment about the x-axis

$$M_x = \int_c^d yw(y)\,dy.$$

and we get

$$\overline{y} = \frac{\int_c^d yw(y)\,dy}{\int_c^d w(y)\,dy} = \frac{M_x}{\text{Area}(R)}$$

$$\overline{x} = \frac{\int_a^b xh(x)\,dx}{\int_a^b h(x)\,dx} = \frac{M_y}{\text{Area}(R)}$$

$$\overline{y} = \frac{\int_c^d yw(y)\,dy}{\int_c^d w(y)\,dy} = \frac{M_x}{\text{Area}(R)}$$

By shell method,

$$2\pi M_x = \int_c^d 2\pi yw(y)\,dy = \text{Volume of } R \text{ rotated around } x\text{-axis.}$$

Similarly,

$$2\pi M_y = \int_a^b 2\pi xh(x)\,dx = \text{Volume of } R \text{ rotated around } y\text{-axis.}$$

So we get *Pappus Theorem*:

$$\text{Volume of } R \text{ rotated around } x\text{-axis} = 2\pi\overline{y}\,\text{Area(R)}$$

Example. Let R be the region under the graph of $y = x^2$ on $[0, 1]$. Compute the centroid of R. Is it inside the region?

We have $h(x) = x^2$, so

$$\overline{x} = \frac{\int_0^1 x \cdot x^2 \, dx}{\int_0^1 x^2 \, dx} = \frac{1/4}{1/3} = \frac{3}{4}$$

For $\overline{y}$, we observe that $w(y) = 1 - \sqrt{y}$, so that

$$\overline{y} = \frac{\int_0^1 y(1 - \sqrt{y})\, dy}{\text{Area}(R)}$$

There is an easier option! The numerator is

$$\frac{1}{2\pi} \cdot (\text{Volume of } R \text{ rotated around } x\text{-axis}) = \frac{1}{2\pi} \cdot \pi \int_0^1 x^4\, dx = \frac{1}{10}$$

and so

$$\overline{y} = \frac{1/10}{1/3} = \frac{3}{10}.$$

To check whether $(\overline{x}, \overline{y})$ is inside R, we check whether $\overline{y} < \overline{x}^2$, which holds, so the answer is yes.

Now rotate the same region R around the line $y = 2x - 3$. What is the resulting volume?

By Pappus, the volume equals

$$2\pi \cdot (\text{distance between } (\overline{x}, \overline{y}) \text{ and the axis}) \cdot \text{Area}(R)$$

The formula for the distance between a line given by $Ax + By + C = 0$ and the point $(\overline{x}, \overline{y})$ is given by the formula

$$\frac{|A\overline{x} + B\overline{y} + C|}{\sqrt{A^2 + B^2}}$$

In our case, the line is $2x - y - 3 = 0$ and so

$$\text{Volume} = 2\pi \cdot \frac{|2 \cdot \frac{3}{4} - \frac{3}{10} - 3|}{\sqrt{4+1}} \cdot \frac{1}{3} = \frac{6\pi}{5\sqrt{5}}$$

Example. Volume of torus; we obtain a torus by rotating a circle of radius r centered at $(0, a)$, $a > r$, around the x-axis.

The centroid of the circle clearly is $(a, 0)$. So by Pappus

$$\text{Volume} = 2\pi a \cdot \pi r^2 = 2\pi^2 a r^2$$