Week 8 Lectures

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We want to compute, say, areas of unbounded regions. Such problems lead to either: integrals over unbouded intervals; or integrals of unbounded functions. We start with unbounded intervals.

8.8. Improper integrals

We define

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

If this limit exists, this integral *converges*, otherwise it *diverges*.

We analogously define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

Also,

$$\int_{-\infty}^{\infty} f(x)\,dx$$

converges if both $\int_0^{\infty} f(x)\,dx$ and $\int_{-\infty}^{0} f(x)\,dx$ converge.

Example. Show that $\int_0^{\infty} e^{-x}\,dx$ converges and compute it.

$$\begin{aligned}\int_0^{\infty} e^{-x}\,dx &= \lim_{b\to\infty}\int_0^{b} e^{-x}\,dx \\ &= \lim_{b\to\infty} -e^{-x}\Big|_{x=0}^{x=b} \\ &= \lim_{b\to\infty}\left(-e^{-b}+1\right) = 1\end{aligned}$$

Example. *The p-integrals*. Determine for which p the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges.

For $p \neq 1$,

$$\begin{aligned}\int_1^\infty x^{-p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx\\ &= \lim_{b\to\infty}\frac{x^{1-p}}{1-p}\Big|_{x=1}^{x=b}\\ &= \lim_{b\to\infty}\frac{b^{1-p}-1}{1-p} = \begin{cases}\frac{1}{p-1} & p>1\\ \infty & p<1\end{cases}\end{aligned}$$

For $p = 1$,

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} \ln x \Big|_{x=1}^{x=b}$$
$$= \lim_{b\to\infty} \ln b = \infty$$

The p integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges exactly when $p > 1$.

How about $\int_7^\infty \frac{1}{x^p}\,dx$?
As there are no problems on $[1,7]$:

$$\int_7^\infty \frac{1}{x^p}\,dx = \int_1^\infty \frac{1}{x^p}\,dx - \int_1^7 \frac{1}{x^p}\,dx,$$

this integral also converges exactly when $p > 1$.

Can we show that $\int_a^\infty f(x)\,dx$ converges without computing it?

Example. $\int_1^\infty \frac{x^5}{x^7+1}\,dx$

Comparison test.

Assume that f, g are two *nonnegative* continuous functions on $[a, \infty)$, where g is much simpler than f.

If we know that $f(x) \leq g(x)$ for all $x \geq a$, and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges as well.

If we know that $f(x) \geq g(x)$ for all $x \geq a$, and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges as well.

It is enough that each inequality holds for $x \geq A$ for some $A > a$.

It is enough that each inequality hold for $x \geq A$ for some $A > a$, and also g can be replaced by g times some positive constant in each inequality.

For example, assume that we want to prove that $\int_1^\infty f(x)\,dx$ converges. We know that $\int_1^\infty g(x)\,dx$ converges, and that $f(x) \leq 5g(x)$ for $x \geq 10$. Then

$$\begin{aligned}\int_1^\infty f(x)\,dx &= \int_1^{10} f(x)\,dx + \int_{10}^\infty f(x)\,dx \\ &\leq \int_1^{10} f(x)\,dx + 5\int_{10}^\infty g(x)\,dx,\end{aligned}$$

which is finite.

Example. $\int_1^\infty \frac{x^5}{x^7+1}\,dx$

We can see that

$$\frac{x^5}{x^7+1} \leq \frac{x^5}{x^7} = \frac{1}{x^2}$$

and we know that $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so we conclude that the integral converges.

The integral $\int_0^\infty \frac{x^5}{x^7+1}\,dx$ then also converges.

Example. $\int_0^\infty e^{-x^2}\,dx$

We can see that for $x \geq 1$

$$e^{-x^2} \leq e^{-x}$$

and we know that $\int_0^\infty e^{-x}\,dx$ converges, so we conclude that the integral converges.

Example. $\displaystyle\int_1^\infty \frac{1+e^{-\sin x}}{x}\,dx$

We can see that for $x \geq 1$

$$\frac{1+e^{-\sin x}}{x} \geq \frac{1}{x}$$

and we know that $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges, so we conclude that the integral diverges.

Limit comparison 1.

Assume that f, g are two nonnegative continuous functions on $[a, \infty)$, where g is much simpler than f.

Assume that we know that $\int_a^\infty g(x)\,dx$ converges. Assume also that we can compute

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

and L is not ∞, but can be 0. The we can conclude that $\int_a^\infty f(x)\,dx$ also converges.

The reason is that there is a large enough A so that $\frac{f(x)}{g(x)} \leq L+1$ for $x \geq A$, which means that $f(x) < (L+1)g(x)$ when $x \geq A$.

Limit comparison 2.

Assume now that we know that $\int_a^\infty g(x)\,dx$ diverges. Assume also that we can compute

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

and L is not 0, but can be ∞. The we can conclude that $\int_a^\infty f(x)\,dx$ also diverges.

Here is the reason. Take $c = L/2$ when L is finite and $c = 1$ when $L = \infty$, so that $c > 0$. We now know that there is a large enough A so that $\dfrac{f(x)}{g(x)} \geq c$ when $x \geq A$, that is, that $f(x) \geq cg(x)$ when $x \geq A$.

Example. $\displaystyle\int_1^{\infty} \frac{\sqrt{x}-1}{x(\sqrt{x}+1)}\,dx$

Let the function inside the integral be $f(x)$. To find a comparison function, multiply out and take the largest powers on top and on bottom. In this case, we get

$$g(x) = \frac{x^{1/2}}{x^{3/2}} = \frac{1}{x}$$

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x\cdot(\sqrt{x}-1)}{x(\sqrt{x}+1)} = 1$$

and so our integral and $\displaystyle\int_1^{\infty} g(x)\,dx = \int_1^{\infty} \frac{1}{x}\,dx$ converge or diverge together.

But we know that $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges, so our integral diverges as well.

This method works for any rational function.

Example. $\displaystyle\int_1^\infty \frac{3x^{5/2} + x^{3/2} + 2022}{x^4 + x^2 + 1}\,dx$

Let the function inside the integral be $f(x)$. In this case, we get

$$g(x) = \frac{x^{5/2}}{x^4} = \frac{1}{x^{3/2}}$$

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{3x^4 + \text{(lower powers)}}{x^4 + \text{(lower powers)}} = 3$$

and so our integral and $\displaystyle\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^{3/2}}\,dx$ converge or diverge together.

But we know that $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges, so our integral converges as well.

8.8. Improper integrals

We now look at the case of unbounded functions on finite intervals.
If f is continuous on $[a, b]$, except at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a+} f(x)\,dx$$

If we have two or more points of discontinuity (or an interior point of discontinuity), we separate integrals so that we have one problem per integral.

Example. *The p-integrals*. Determine for which p the integral $\int_0^1 \frac{1}{x^p}\,dx$ converges.

For $p \neq 1$,

$$\begin{aligned}\int_0^1 x^{-p}\,dx &= \lim_{c\to 0+}\int_c^1 x^{-p}\,dx \\ &= \lim_{c\to 0+} \frac{x^{1-p}}{1-p}\Big|_{x=c}^{x=1} \\ &= \lim_{c\to 0+} \frac{1-c^{1-p}}{1-p} = \begin{cases}\frac{1}{1-p} & p<1 \\ \infty & p>1\end{cases}\end{aligned}$$

For $p = 1$,

$$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{x}\,dx$$
$$= \lim_{c\to 0+} \ln x \Big|_{x=c}^{x=1}$$
$$= \lim_{c\to 0+} -\ln c = \infty$$

The p integral $\int_0^1 \frac{1}{x^p}\,dx$ converges exactly when $p < 1$.

We conclude that the integral

$$\int_0^\infty x^{-p}\,dx$$

diverges for all p.

Example. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

We proceed by definition

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{c\to 1-} \int_0^c \frac{1}{\sqrt{1-x^2}}\,dx \\ &= \lim_{c\to 1-} \arcsin x \Big|_{x=0}^{x=c} \\ &= \lim_{c\to 1-} \arcsin c = \frac{\pi}{2},\end{aligned}$$

so the integral converges.

Here is a review problem.

Example. Determine convergence or divergence of
$$\int_1^\infty \frac{x + \ln x + 7}{x^3 \ln x + x + 5}\, dx$$

Let the function inside the integral be $f(x)$. Try this for a comparison function:

$$g(x) = \frac{x}{x^3} = \frac{1}{x^2}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

$$= \lim_{x\to\infty} \frac{x^2(x + \ln x + 7)}{x^3 \ln x + x + 5} = \lim_{x\to\infty} \frac{x^3}{x^3 \ln x} = \lim_{x\to\infty} \frac{1}{\ln x} = 0$$

The limit is 0. But this is good enough! We know that $\int_1^\infty g(x)\, dx = \int_1^\infty \frac{1}{x^2}\, dx$ converges, so our integral converges as well.