Week 9 Lectures

Janko Gravner

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11.1–2. Parametric curves

Position of a particle is the plane is given by $x = f(t)$, and $y = g(t)$. This means that the position (x, y) changes with changing t (which is usually, but not always, thought of as time). If $t \in [a, b]$, then $(f(a), g(a))$ is the *initial* point and $(f(b), g(b))$ is the *final* (or *terminal*) point.

Example. Throw a stone horizontally, with velocity 2 (m/s), from height 4 (m). Ignoring the air resistance, and assuming $g = 10$ (m/s^2), write the parametric equation for the stone's trajectory. Describe the curve on which the particle is moving by writing the y-coordinate y as a function of the x-coordinate x.

Center the coordinate system so that the initial point is $(0, 4)$. At time t, the position is then given by

$$x = 2t$$
$$y = 4 - \frac{g}{2}t^2 = 4 - 5t^2$$

Assume the stone stops when it hits the ground. This happens when $y = 0$, so $t = 2/\sqrt{5}$, and the final point is $(4/\sqrt{5}, 0)$.

If we solve $x = 2t$ for x, we get $t = x/2$ and then

$$y = 4 - \frac{5}{4}x^2,$$

a parabola.

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Example. $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le \pi$. Describe the motion of the particle.

The particle moves counterclockwise on the circle of radius 3 center at the origin, with initial point $(3, 0)$ and final point $(-3, 0)$.

Example. $x = \sqrt{t}$, $y = t + 1$, $0 \le t \le \pi$. Describe the curve on which the particle is moving and the direction of motion.

Solve $x = \sqrt{t}$ for t to get $t = x^2$ and so $y = x^2 + 1$. The particle is moving on the parabola $y = x^2 + 1$. As x increases with t, the particle moves rightwards.

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We want to do calculus with parametric curves. We begin with tangent lines.

Assume that the curve is given by $x = f(t)$, $y = g(t)$, where both f and g are differentiable. Compute the tangent at a point $(f(t_0), g(t_0))$.
The slope of the tangent is given by $\frac{dy}{dx}$, imagining that we can write y as a function of x. By the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and so, provided $\frac{dx}{dt} \neq 0$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

into which we can plug in $t = t_0$.

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This way, we can compute the second derivative

$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$$

The curve given by $y = f(x)$ can be interpreted as a parametric curve: $x = t$, $y = f(t)$, in which particle moves on the curve so that its x coordinate moves at unit speed.

Example. Consider the curve $x = t^3 + t^2 + 1$, $y = t^4 + t^3$, $t \geq 0$. Find the tangent at $t = 1$.

At $t = 1$, the point on the curve is $(3, 2)$.

Moreover

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3 + 3t^2}{3t^2 + 2t}$$

and we plug in $t = 1$ to get the slope $7/5$. The tangent line is:

$$y - 2 = \frac{7}{5}(x - 3)$$

Example. Consider the curve given parametrically by $x = t^4 + 2$, $y = t^3 + t$, $t \geq 0$. Analyze monotonicity and concavity properties of this curve and describe the motion of the particle. Roughly sketch this curve.

We have $\frac{dx}{dt} = 4t^3 \geq 0$, $\frac{dy}{dt} = 3t^2 + 1 \geq 0$. So, x and y are both increasing in t, and

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 + 1}{4t^3} = \frac{3}{4}t^{-1} + \frac{1}{4}t^{-3} > 0$$

Further,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{-\frac{3}{4}t^{-2} - \frac{3}{4}t^{-4}}{4t^3},$$

which is always negative so that the curve is concave down.

A curve is given prametrically by $x = f(t)$, and $y = g(t)$. How can we compute the area under this curve, between $t = t_0$ and $t = t_1$? *Assuming that x increases with t*, and $y > 0$ we divide the region into thin strips between positions at t and $t + \Delta t$. The strips have height $y = g(t)$ and width $\Delta x = f(t + \Delta t) - f(t) \approx f'(t)\Delta t$. The area of the strip is the approximately $g(t)f'(t)\Delta t$, which leads to the integral

$$\text{Area} = \int_{t_0}^{t_1} g(t)f'(t)\,dt = \int_{t_0}^{t_1} y\frac{dx}{dt}\,dt = \int_{t_0}^{t_1} y\,dx$$

(If x decreases with t, replace $\frac{dx}{dt}$ with $-\frac{dx}{dt}$.)

Example. Again, consider the curve given parametrically by $x = t^4 + 2$, $y = t^3 + t$, $t \geq 0$. Compute the area below the curve between $t = 0$ and $t = 1$.

We know that x coordinate is increasing and $y > 0$ for t in $[0, 1]$ so that the answer is

$$\int_0^1 (t^3 + t) \cdot 4t^3 \, dt = \frac{4}{7} + \frac{4}{5} = \frac{48}{35}$$

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A curve is given prametrically by $x = f(t)$, and $y = g(t)$. How can we compute the arc length this curve, between $t = t_0$ and $t = t_1$? We divide the region into pieces between positions at t and $t + \Delta t$. A small piece has length has length approximated by Δs, where

$$(\Delta s)^2 = (\Delta x)^2+(\Delta y)^2 \approx f'(t)^2\Delta t^2+g'(t)^2\Delta t^2 = (f'(t)^2+g'(t)^2)\Delta t^2$$

and so $\Delta s \approx \sqrt{f'(t)^2 + g'(t)^2}\Delta t$,

$$\text{Arc Length} = \int_{t_0}^{t_1} \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

$$= \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t_0}^{t_1} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_{t_0}^{t_1} ds$$

Example. Again, the curve is $x = t^4 + 2$, $y = t^3 + t$, $t \geq 0$. Write down the integral for the arc length between $t = 0$ and $t = 1$.

The answer is

$$\int_0^1 \sqrt{(4t^3)^2 + (3t^2 + 1)^2}\, dt$$

Example. The motion of a particle is described by the parametric curve $x = t^3 + 1$, $y = t^2 - t + 1$, $t \geq 0$.
(a) Using first and second derivatives, analyze and roughly sketch this curve. Describe the motion of the particle.
(b) Compute the area below the curve between the initial point and the inflection point.
(c) Rotate the region below the curve around the x-axis and compute the volume.
(d) Rotate the region below the curve around the y-axis and compute the volume.
(e) Compute the arc length of this curve.
(f) Rotate the curve around the x-axis and compute the surface area.
(g) Rotate the curve around the y-axis and compute the surface area.

11.1–2. Parametric curves: review example

(a) Using first and second derivatives, analyze and roughly sketch this curve. Describe the motion of the particle.

We have $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t - 1$, so that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-1}{3t^2} = \frac{2}{3}t^{-1} - \frac{1}{3}t^{-2}$$

which is positive when $t > 1/2$ and negative when $t < 1/2$. The curve has a global minimum at $t = 1/2$, where $(x, y) = (9/8, 3/4)$.

More precisely: $\frac{dx}{dt} = 3t^2$ is always positive, the x coordinate always increases, the particle is moving rightwards; and $\frac{dy}{dt} = 2t - 1$ so that $\frac{dy}{dt} < 0$, y coordinate decreases and the particle is moving downwards for $t < 1/2$, and upwards for $t > 1/2$.

11.1–2. Parametric curves: review example

Further,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}\left(\frac{2}{3}t^{-1} - \frac{1}{3}t^{-2}\right)}{3t^2} \\ &= \frac{-\frac{2}{3}t^{-2} + \frac{2}{3}t^{-3}}{3t^2} \\ &= -\frac{2}{9}t^{-5}(t-1)\end{aligned}$$

which is positive when $t < 1$ (so that the curve is concave down) and negative when $t > 1$ (so that the curve is concave up). The inflection point is when $t = 1$, at $(x, y) = (2, 1)$.

(b) Compute the area below the curve between the initial point and the inflection point.

Those are the points given by $t = 0$ and $t = 1$, so that the answer is

$$\int_0^1 y\,dx = \int_0^1 (t^2 - t + 1) \cdot 3t^2\,dt$$

(c) Rotate the region below the curve around the x-axis and compute the volume.

The volume is given by

$$\text{Volume} = \pi \int_0^1 y^2\,dx = \pi \int_0^1 (t^2 - t + 1)^2 3t^2\,dt$$

(d) Rotate the region below the curve around the y-axis and compute the volume.

The volume is given by

$$\text{Volume} = 2\pi \int_0^1 xy\,dx = \pi \int_0^1 (t^3 + 1)(t^2 - t + 1)3t^2\,dt$$

(e) Compute the arc length of this curve.

The arc length is given by

$$\text{Arc Length} = \int_0^1 \sqrt{(3t^2)^2 + (2t-1)^2}\, dt$$

(f) Rotate the curve around the x-axis and compute the surface area.

The surface area is given by

$$\text{Surface Area} = 2\pi \int_0^1 y\, ds = 2\pi \int_0^1 (t^2 - t + 1)\sqrt{(3t^2)^2 + (2t-1)^2}\, dt$$

(g) Rotate the curve around the y-axis and compute the surface area.

The surface area is given by

$$\text{Surface Area} = 2\pi \int_0^1 x\, ds = 2\pi \int_0^1 (t^3+1)\sqrt{(3t^2)^2 + (2t-1)^2}\, dt$$

Optional fun problem.

From A. C. Doyle, *The Adventure of the Priory School* (1904):

We had come on a small black ribbon of pathway. In the middle of it, clearly marked on the sodden soil, was the track of a bicycle.
"Hurrah!"I cried. "We have it."
But Holmes was shaking his head, and his face was puzzled and expectant rather than joyous.
"A bicycle, certainly, but not the bicycle" said he."I am familiar with forty-two different impressions left by tires. This, as you perceive, is a Dunlop, with a patch upon the outer cover. Heidegger's tires were Palmer's, leaving longitudinal stripes. Aveling, the mathematical master, was sure upon the point. Therefore, it is not Heidegger's track."
"The boy's, then?"
"Possibly, if we could prove a bicycle to have been in his possession. But this we have utterly failed to do.

This track, as you perceive, was made by a rider who was going from the direction of the school."
"Or towards it?"
"No, no, my dear Watson. The more deeply sunk impression is, of course, the hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school. It may or may not be connected with our inquiry, but we will follow it backwards before we go any farther."
We did so, and at the end of a few hundred yards lost the tracks as we emerged from the boggy portion of the moor. Following the path backwards, we picked out another spot, where a spring trickled across it. Here, once again, was the mark of the bicycle, though nearly obliterated by the hoofs of cows. After that there was no sign, but the path ran right on into Ragged Shaw, the wood which backed on to the school.

From this wood the cycle must have emerged. Holmes sat down on a boulder and rested his chin in his hands. I had smoked two cigarettes before he moved. “Well, well,” said he, at last. “It is, of course, possible that a cunning man might change the tires of his bicycle in order to leave unfamiliar tracks. A criminal who was capable of such a thought is a man whom I should be proud to do business with. We will leave this question undecided and hark back to our morass again, for we have left a good deal unexplored.”

We continued our systematic survey of the edge of the sodden portion of the moor, and soon our perseverance was gloriously rewarded. Right across the lower part of the bog lay a miry path. Holmes gave a cry of delight as he approached it. An impression like a fine bundle of telegraph wires ran down the centre of it. It was the Palmer tires.

“Here is Herr Heidegger, sure enough!” cried Holmes, exultantly. “My reasoning seems to have been pretty sound, Watson.”

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