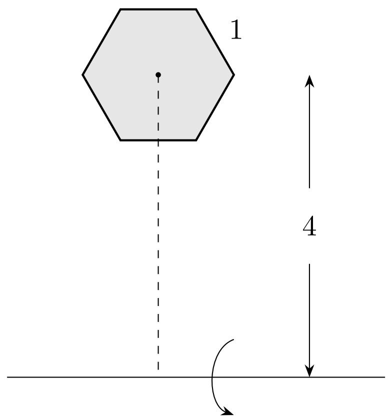
Math 21B, Winter 2023.

Discussion Problems 8 (Thu., Nov. 30)

1. Determine the center of mass of the region bounded by $y = 2\sin(2x)$, $0 \le x \le \pi/2$ and $y = 0$.

2. (a) Determine the center of mass of the region R bounded by $y = x^3$ and $y = \sqrt{x}$. (b) Rotate R around the line $y = (3/2)x + 1$. Compute the volume of the resulting solid.

3. A regular hexagon (with side length 1) is rotated around an axis as in the picture. Determine the volume of the resulting solid. (Minimize your work!)



4. Determine whether the following integrals converge or diverge.

(a) $\displaystyle\int_0^\infty \frac{e^{-x}}{1+x^2}\,dx$ (b) $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ (c) $\displaystyle\int_0^3 \frac{1}{(x-3)^2}\,dx$

(d) $\displaystyle\int_0^\infty \frac{x^{9/2}+3x^{7/2}}{x^6+x^3+x^{3/2}+1}\,dx$ (e) $\displaystyle\int_0^5 \frac{1}{\sqrt{25-x^2}}\,dx$ (f) $\displaystyle\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$

(g) $\displaystyle\int_{10}^\infty \frac{x^5\arctan x}{x^7+1}\,dx$ (h) $\displaystyle\int_{10}^\infty \frac{x^5\ln x}{x^7+1}\,dx$

Here is a clarification on the computation of the centroid of the region R bounded between graphs $y = f(x)$ and $y = g(x)$ on $[a, b]$, where $f(x) \ge g(x)$ (so f is the top function) but one or both functions may cross the x-axis. We compute

$$\begin{aligned}
\text{Area}(R) &= \int_a^b (f(x) - g(x))\,dx \\
M_y &= \int_a^b x(f(x) - g(x))\,dx \\
M_x &= \frac{1}{2}\int_a^b (f(x)^2 - g(x)^2)\,dx
\end{aligned} \tag{1}$$

Then

$$\overline{x} = \frac{M_y}{\text{Area}(R)}, \quad \overline{y} = \frac{M_x}{\text{Area}(R)} \tag{2}$$

So we use the formula for M_x as if both graphs are above the x-axis. The derivation below is for those interested; you are not required to know it. (The book has a different derivation.)

Let k be a number large enough so that both graphs are above the x-axis, Adding k to both graphs translates the region by k upwards, and so the y-coordinate $\overline{y}'$ of the centroid of the new region R' is also increased by k. Now, we know from lecture that

$$\overline{y}' = \frac{\frac{1}{2\pi}(\text{Volume of } R' \text{ rotated around } x\text{-axis})}{\text{Area}(R)}$$

and so

$$\begin{aligned}
\overline{y} = \overline{y}' - k &= \frac{\frac{1}{2\pi}\pi \int_a^b \left((f(x)+k)^2 - (g(x)+k)^2\right)\, dx}{\int_a^b (f(x)-g(x))\, dx} - k \\
&= \frac{\frac{1}{2}\int_a^b \left((f(x)+k)^2 - (g(x)+k)^2\right)\, dx - k\int_a^b (f(x)-g(x))\, dx}{\int_a^b (f(x)-g(x))\, dx} \\
&= \frac{\frac{1}{2}\int_a^b \left((f(x)^2+2kf(x)+k^2) - (g(x)^2+2kg(x)+k^2)\right)\, dx - k\int_a^b (f(x)-g(x))\, dx}{\int_a^b (f(x)-g(x))\, dx} \\
&= \frac{\frac{1}{2}\int_a^b \left(f(x)^2 - g(x)^2\right)\, dx}{\int_a^b (f(x)-g(x))\, dx}
\end{aligned}$$

5. Apply formulas (1) and (2) to set up the integrals for the coordinates of the centroid of the region R bounded by the graphs of $y = x^2 - x$ and $y = 5x - 2x^2$.