Math 21B, Winter 2022.
March. 14, 2022.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________

ID#: ______________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 10 pages (including this one) with 8 problems.

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

distance between (x_1, y_1) and $Ax + By + C = 0$: $\dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B)), \quad \sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B)),$

$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B)),$

$\sin^2 A = \frac{1}{2}(1 - \cos(2A)), \quad \cos^2 A = \frac{1}{2}(1 + \cos(2A)).$

1. Let $a_n = \sum_{i=1}^{n} \frac{2n+i}{n(3n+i)}$.

(a) Compute: $a = \lim_{n\to\infty} a_n$.

the Riemann sum with right endpoints as eval. pts.

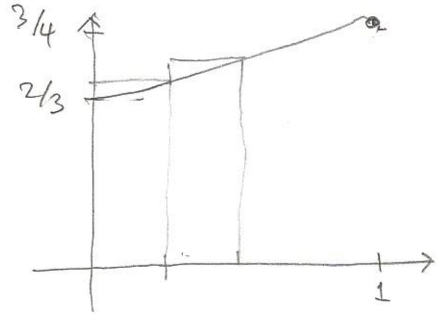
$$a_n = \sum_{i=1}^{n} \frac{2+\frac{i}{n}}{3+\frac{i}{n}} \cdot \frac{1}{n} = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} \to \int_0^1 f(x)\,dx$$

$$= \int_0^1 \frac{2+x}{3+x}\,dx = \int_0^1 \left(1 - \frac{1}{3+x}\right)dx$$

$$= x - \ln(3+x)\Big|_0^1$$

$$= 1 - \ln 4 + \ln 3$$

$$= \underline{\underline{1 + \ln 3 - 2\ln 2}}$$

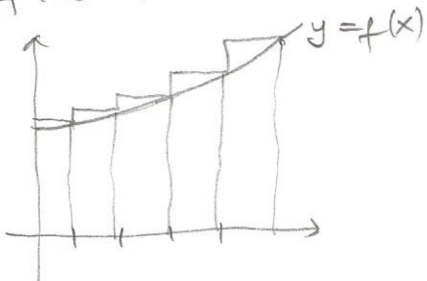


$$f(x) = \frac{2+x}{3+x} = 1 - \frac{1}{3+x}$$

$$f'(x) = \frac{1}{(3+x)^2} > 0$$

(b) Is a_5 larger or smaller than a? Explain fully.

As the Riemann sum uses right endpts. and the function is increasing, $\underline{a_5 > a}$.



a = area under the graph

a_5 = sum of the areas of rectangles

2. Compute the following indefinite integrals.

(a) $\int \frac{3x-1}{(x+1)(x^2+1)}\, dx = (*)$

$$\frac{3x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$* (x+1),\ x=-1:\quad A=-2$

$* x,\ x\to\infty:\quad 0 = A+B,\quad B=2$

$x=0:\quad -1 = A+C,\quad C=-1$

$$(*) = \int\left(\frac{-2}{x+1} + \frac{2x-1}{x^2+1}\right)dx$$

$$= \underline{-2\ln|x+1| + \ln(x^2+1) - \arctan x + C}$$

(b) $\int \frac{1}{\sqrt{x}} \ln(x)\, dx$

$u = \ln x \qquad du = \frac{1}{x}dx$

$dv = \frac{1}{\sqrt{x}}dx \qquad v = 2\sqrt{x}$

$$= 2\sqrt{x}\ln x - \int \frac{2}{\sqrt{x}}\,dx = \underline{2\sqrt{x}\ln x - 4\sqrt{x} + C}$$

3. Determine whether the two integrals below converge or diverge.

(a) $\displaystyle\int_1^{\infty} \underbrace{\frac{2x+3x^{1/3}+4}{\sqrt{x}(x^2+2x+3)}}_{\substack{\| \\ f(x)}} dx$

$$g(x) = \frac{1}{x^{3/2}}$$

$$\frac{f(x)}{g(x)} = \frac{2x^{5/2} + (\text{lower powers})}{x^{5/2} + (\text{—''—})} \to 2$$

As $\int_1^{\infty} g(x)\,dx$ converges (p-integral with $p = 3/2 > 1$),

$\int_1^{\infty} f(x)\,dx$ <u>converges</u>.

(b) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\, dx$

$$= \lim_{a\to 0} \int_a^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\, dx = \lim_{a \to 0} \int_{\sin a}^{1} \frac{1}{\sqrt{u}}\, du$$

$u = \sin x$, $du = \cos x\, dx$,

x	u
a	$\sin a$
$\pi/2$	1

$$= \lim_{a\to 0} 2\sqrt{u}\,\Big|_{\sin a}^{1} = \lim_{a \to 0} \left(2 - 2\sqrt{\sin a}\right) = \underline{\underline{2}}$$

<u>Converges</u>

4. Consider the functions

$$f(x) = \int_0^x \sqrt{9+t+t^3}\,dt, \qquad g(x) = \int_0^x t\sqrt{9+t^3}\,dt.$$

(a) Compute $\lim_{x\to 0} \frac{f(x)}{x}$.

$$\underset{\left(\frac{0}{0}\right)}{} \underset{\text{(L'Hopital)}}{=} \lim_{x\to 0} \frac{\sqrt{9+x+x^3}}{1} = \underline{\underline{3}}$$

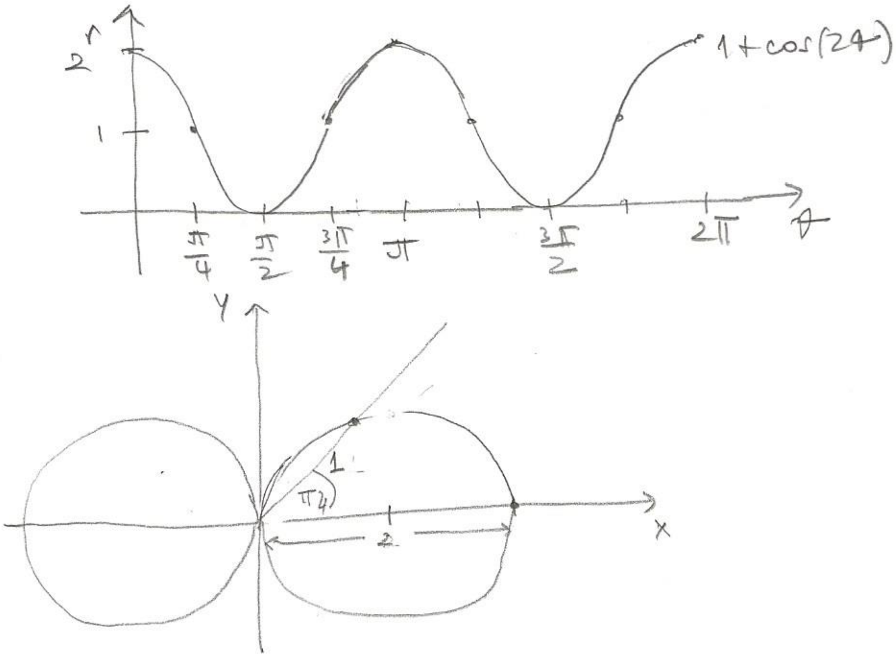
(b) Compute $\lim_{x\to 0} \frac{f(x)^2}{g(x)}$.

$$\underset{\left(\frac{0}{0}\right)}{} \underset{\text{(L'Hopital)}}{=} \lim_{x\to 0} \frac{2f(x)f'(x)}{g'(x)}$$

$$= \lim_{x\to 0} \frac{2\underbrace{\frac{f(x)}{x}}_{3 \text{ (by (a))}} \cdot \sqrt{9+x+x^3}}{\sqrt{9+x^3}}$$

$$= \frac{2\cdot 3\cdot 3}{3} = \underline{\underline{6}}$$

5. Consider the curve $r = 1 + \cos(2\theta)$ in polar coordinates.
(a) Sketch the curve.



(b) Compute the area this curve encloses.

$$4 \cdot \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2\theta)^2 \, d\theta$$

$$= 2 \int_0^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta) \, d\theta$$

$$= 2 \int_0^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right) d\theta$$

$$= 2 \int_0^{\pi/2} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= 2 \cdot \left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]\Bigg|_0^{\pi/2}$$

$$= \underline{\underline{\frac{3\pi}{2}}}$$

6. Consider the curve given parametrically by $x = t^3 + 3t^2 + 9t$ and $y = t^2 + t$, $t \geq 0$.

(a) Find the equation of the tangent to the curve at $t = 1$. (You may leave the equation in the point-slope form.)

$$\frac{dx}{dt} = 3t^2 + 6t + 9$$

$$\frac{dy}{dt} = 2t + 1$$

$$\frac{dy}{dx} = \frac{2t+1}{3t^2+6t+9} = \frac{1}{3} \cdot \frac{2t+1}{t^2+2t+3}$$

At $t=1$:

pt. $(13, 2)$

slope: $\frac{1}{6}$

line $y - 2 = \frac{1}{6}(x - 13)$

(b) Determine where this curve is concave up and where it is concave down. Sketch this curve, indicating clearly any inflection points.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{1}{9(t^2+2t+3)} \cdot \frac{2\cdot(t^2+2t+3) - (2t+2)(2t+1)}{(t^2+2t+3)^2}$$

$$= \frac{2\left[t^2+2t+3 - 2t^2 - t - 2t - 1\right]}{9(t^2+2t+3)^3} = \frac{-2}{9} \frac{t^2+t-2}{(t^2+2t+3)^3}$$

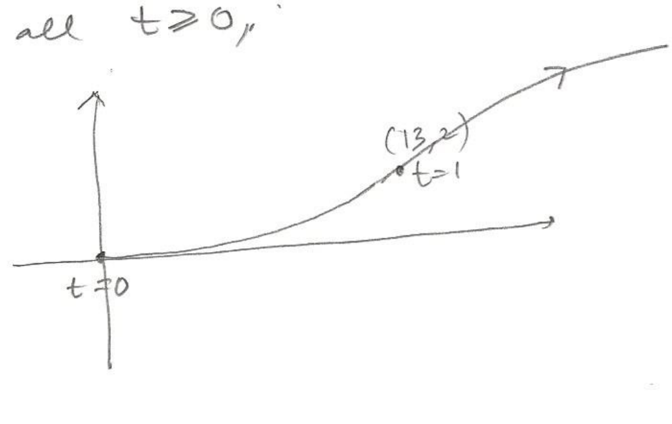
$t^2 + t - 2 = (t+2)(t-1)$

infl. pt. $t = 1$, $(x,y) = (13,2)$

$\frac{d^2y}{dx^2} > 0$ if $t < 1$, concave up

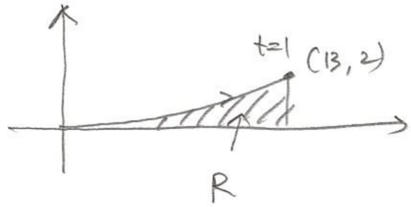
$\frac{d^2y}{dx^2} < 0$ if $t > 1$, concave down

$\frac{dx}{dt} > 0$ and $\frac{dy}{dt} > 0$ for all $t \geq 0$.



Problem 6, continued.

(c) Let R be the region bounded by the curve, the coordinate axes, and the line $x = 13$. Set up, but do not evaluate, the integral for the area of the region R. (Recall from (a) the point on the curve when $t = 1$.)



$$\text{Area}(R) = \int_0^1 y\,dx$$

$$= \int_0^1 (t^2+t)\,(3t^2+6t+9)\,dt$$

(d) Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating the region R (which is the same as in (c)) around the y–axis.

$$\text{Vol.} = 2\pi \int_0^1 x\,y\,dx = 2\pi \int_0^1 (t^3+3t^2+9t)\,(t^2+t) \cdot (3t^2+6t+9)\,dt$$

(e) Rotate the the piece of the curve between $t = 2$ and $t = 3$ around the line $y = 3$. Set up, but do not evaluate, the integral for the surface area of the resulting surface. (Explain why the line does not intersect the piece of the curve.)

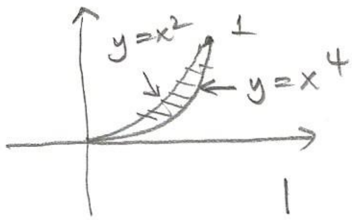
The y-coordinate is increasing between $t=2$ and $t=3$.
The lowest point on the curve is at $t=2$,
when $y=6$.

$$\text{Surface Area} = 2\pi \int_2^3 (y-3)\,ds$$

$$= 2\pi \int_2^3 (t^2+t-3)\,\sqrt{(3t^2+6t+9)^2+(2t+1)^2}\,dt$$

7. The region R lies in the first quadrant and is bounded by the curves $y = x^2$ and $y = x^4$. Compute the quantities below.

(a) Coordinates of the centroid of R. Express your answer in simple fractions.



The curves intersect only at $x = 1$.
$x^2 \geq x^4$ when x is in $[0, 1]$.

$$\text{Area}(R) = \int_0^1 (x^2 - x^4)\,dx = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

$$M_y = \int_0^1 x(x^2 - x^4)\,dx = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

$$M_x = \frac{1}{2}\int_0^1 (x^4 - x^8)\,dx = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{2}{45}$$

$$\underline{\bar{x}} = \frac{M_y}{\text{Area}(A)} = \frac{1/12}{2/15} = \underline{\frac{5}{8}} \qquad \underline{\bar{y}} = \frac{M_x}{\text{Area}(A)} = \frac{\frac{2}{45}}{2/15} = \underline{\frac{1}{3}}$$

(b) The volume of the solid obtained by rotating R around the line $2x - 3y - 2 = 0$.

Pappus:

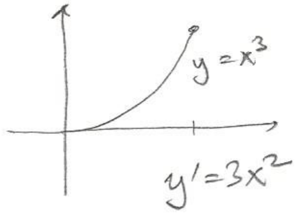
$$\text{Vol.} = 2\pi \cdot \text{dist}((\bar{x}, \bar{y}), \text{line}) \cdot \text{Area}(R)$$

$$= 2\pi \frac{|2\bar{x} - 3\bar{y} - 2|}{\sqrt{4+9}} \cdot \frac{2}{15}$$

$$= \frac{4\pi}{15} \frac{\left|\frac{5}{4} - 1 - 2\right|}{\sqrt{13}} = \frac{7\pi}{15\sqrt{13}}$$

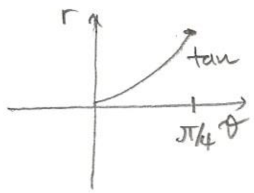
8. Set up, but do not evaluate, the integrals for the arc length of the following curves. Also sketch roughly each of the curves (no concavity analysis necessary).

(a) $y = x^3$, $0 \le x \le 1$, in Cartesian coordinates.

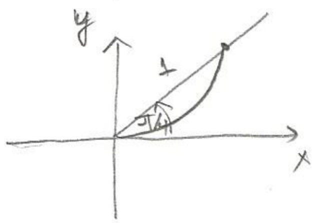


$$A.L. = \int_0^1 \sqrt{1 + 9x^4}\, dx$$

(b) $r = \tan\theta$, $0 \le \theta \le \pi/4$, in polar coordinates.



$$\frac{dr}{dt} = \frac{1}{\cos^2\theta}$$

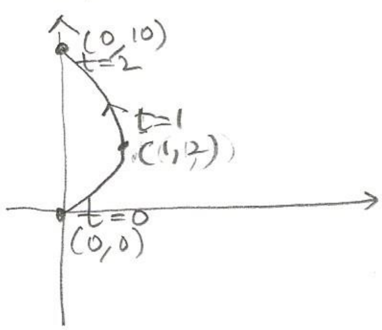


$$A.L. = \int_0^{\pi/4} \sqrt{\tan^2\theta + \frac{1}{\cos^4\theta}}\, d\theta$$

(c) $x = 2t - t^2$, $y = t^3 + t$, $0 \le t \le 2$, given parametrically.

$$\frac{dx}{dt} = 2 - 2t = 2(1-t) \quad \begin{matrix} \ge 0 & \text{if } t < 1 \\ < 0 & \text{if } t > 1 \end{matrix}$$

$$\frac{dy}{dt} = 3t^2 + 1 > 0$$



$$A.L. = \int_0^2 \sqrt{(2-2t)^2 + (3t^2+1)^2}\, dt$$