Math 21B, Fall 2023.
Dec. 12, 2023.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam.

Make sure that you have a total of 10 pages (including this one) with 8 problems.

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

distance between (x_1, y_1) and $Ax + By + C = 0$: $\dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$, $\quad \sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$,

$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$,

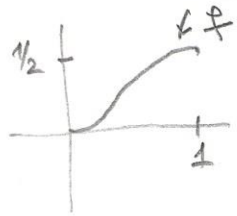
$\sin^2 A = \frac{1}{2}(1 - \cos(2A))$, $\quad \cos^2 A = \frac{1}{2}(1 + \cos(2A))$.

1. Let $a_n = \sum_{i=1}^{n} \frac{i^2}{n(n^2+i^2)}$. $= \sum_{i=1}^{n} \frac{\left(\frac{i}{n}\right)^2}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n}$

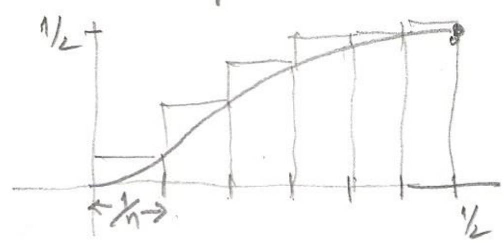
(a) Compute: $a = \lim_{n\to\infty} a_n$.

$$f(x) = \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$$

$$f'(x) = \frac{1}{(1+x^2)^2} \cdot 2x > 0 \text{ for } x > 0,\ f \text{ is } \underline{\text{increasing}} \text{ for } x \geq 0,$$



a_n is the Riemann sum for $y = f(x)$ on $[0,1]$, for partition into n intervals of equal length and evaluation pts. at the right endpts:



So $\lim a_n = \int_0^1 \frac{x^2}{1+x^2}\,dx$

$= \int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx$

$= 1 - \arctan x \Big|_0^1$

$= \underline{\underline{1 - \frac{\pi}{4}}}$

(b) Is a_6 larger or smaller than a? Justify your answer.

Larger, as f is increasing.
(see picture above)

2. Compute the following indefinite integrals.

(a) $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$*(x-1)^2,\quad x=1 : \quad B = 2$$

$$*(x+1),\quad x=-1 : \quad C = -1$$

$$*\,x,\quad x \to \infty : \quad 0 = A + C,\quad A = 1$$

$$= \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right) dx$$

$$= \underline{\ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C}$$

(b) $\displaystyle\int x^3 \cos(2x^2)\,dx \quad = \int x \cdot x^2 \cos(2x^2)\,dx$

$$t = 2x^2 \qquad dt = 4x\,dx \qquad x\,dx = \frac{1}{4}dt$$

$$= \frac{1}{8}\int t \cos t\,dt$$

$$u = t \qquad dv = \cos t\,dt$$

$$du = dt \qquad v = \sin t$$

$$= \frac{1}{8}\left[t\sin t - \int \sin t\,dt \right]$$

$$= \frac{1}{8}\left[t\sin t + \cos t \right] + C$$

$$= \underline{\frac{1}{8}\left[2x^2 \sin(2x^2) + \cos(2x^2) \right] + C}$$

3. Determine whether the two improper integrals below converge or diverge.

(a) $\int_1^e \frac{1}{x \cdot (\ln x)^2}\, dx$

$$= \lim_{c \to 1^+} \int_c^e \frac{1}{x(\ln x)^2}\, dx$$

$$\int \frac{1}{x(\ln x)^2} dx = -\frac{1}{\ln x} + C$$

$$= \lim_{c \to 1^+} \left[-\frac{1}{\ln x}\right]_c^e$$

$$= \lim_{c \to 1^+} \left[-1 + \frac{1}{\ln c}\right] = \infty$$

The integral <u>diverges</u>.

(b) $\int_1^\infty \underbrace{\sqrt{x} \cdot \frac{x^2+2x+3}{4x^4+5x^3+6x^2}}_{f(x)=}\, dx$

$$g(x) = \frac{\sqrt{x} \cdot x^2}{x^4} = \frac{1}{x^{3/2}}$$

we know $\int^\infty g(x)$ converges as its p-integral with $p>1$.

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} x^{3/2} \cdot f(x)$$

$$= \lim_{x \to \infty} \frac{x^4 + \text{L.O.T.}}{4x^4 + \text{L.O.T.}} = \frac{1}{4}$$

By limit comparison, $\int_1^\infty f(x)\,dx$ <u>converges</u>

4. Consider the function

$$g(x) = \int_0^x t \cdot e^{t^3}\, dt.$$

(a) Compute $\lim_{x \to 0} \frac{g(x)}{x^2}$.

$$= \lim_{x \to 0} \frac{\int_0^x t e^{t^3} dt}{x^2} \stackrel{(L'H.)}{=} \lim_{x \to 0} \frac{\cancel{x} e^{x^3}}{2\cancel{x}} = \underline{\underline{\frac{1}{2}}}$$

$\left(\frac{0}{0}\right)$

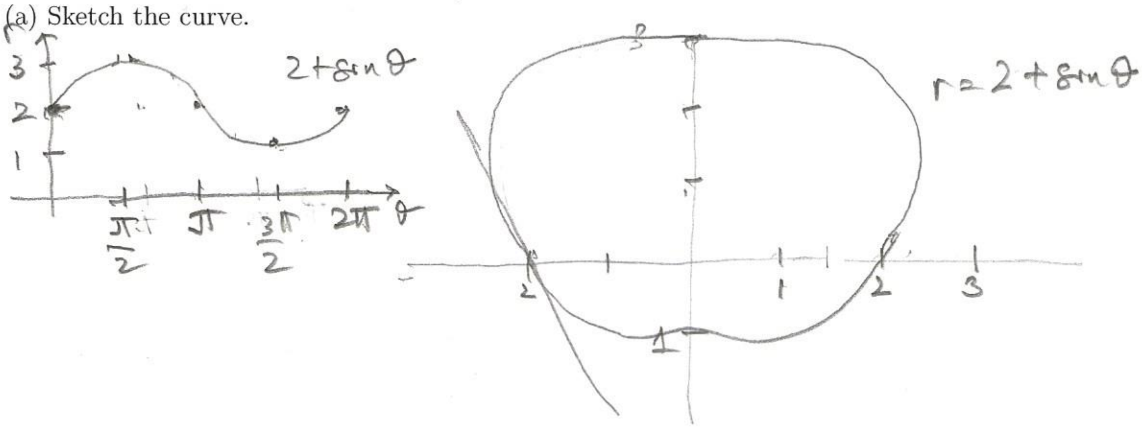
(b) Compute $\lim_{x \to 0} \frac{g(x)}{g(3x)}$. $\left(\frac{0}{0}\right)$

$$\stackrel{(L'H.)}{=} \lim_{x \to 0} \frac{g'(x)}{3g'(3x)} = \lim_{x \to 0} \frac{\cancel{x} e^{x^3}}{3 \cdot (3\cancel{x})\, e^{(3x)^3}}$$

$$= \underline{\underline{\frac{1}{9}}}$$

5. Consider the curve $r = 2 + \sin\theta$ in polar coordinates.

(a) Sketch the curve.



(b) Find the slope of the tangent to the curve at $\theta = \pi$. Make sure the slope is reflected in your picture in (a).

$$x = (2+\sin\theta)\cos\theta \qquad \frac{dx}{d\theta} = \cos\theta\cos\theta - (2+\sin\theta)\sin\theta$$

$$y = (2+\sin\theta)\sin\theta \qquad \frac{dy}{d\theta} = \cos\theta\sin\theta + (2+\sin\theta)\cos\theta$$

at $\theta = \pi$; $\frac{dx}{d\theta} = 1$, $\frac{dy}{d\theta} = -2$

$$\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -2$$

(c) Compute the area this curve encloses.

$$\frac{1}{2}\int_0^{2\pi} (2+\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (4 + 4\sin\theta + \sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 + \cancel{4\sin\theta} + \frac{1}{2} - \cancel{\frac{1}{2}\cos(2\theta)}\right) d\theta$$

these two integrals vanish

$$= \frac{1}{2} \cdot \frac{9}{2} \cdot 2\pi$$

$$= \underline{\underline{\frac{9}{2}\pi}}$$

$$\frac{dx}{dt} = 12t^2 + 3 \qquad \frac{dy}{dt} = 2t$$

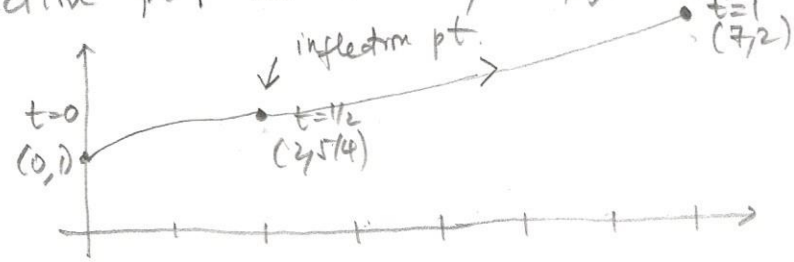
6. Consider the curve given parametrically by $x = 4t^3 + 3t$ and $y = t^2 + 1$, $0 \leq t \leq 1$. Let R be the region bounded by the curve, the coordinate axes, and the line $x = 7$.

(a) Determine where this curve is concave up and concave down. Sketch this curve, indicating clearly any inflection points.

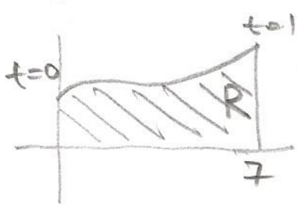
$$\frac{dy}{dx} = \frac{2t}{12t^2+3} \geq 0 \quad \text{on } [0,1]$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{2(12t^2+3) - 2t \cdot 24t}{(12t^2+3)^2}}{12t^2+3} = \frac{-24t^2+6}{(12t^2+3)^3} = \frac{-6(4t^2-1)}{(12t^2+3)^3}$$

Inflection pt. at $t = 1/2$, $(x,y) = (2, 5/4)$



(b) Compute the area of R.



$$\text{Area}(R) = \int_0^1 (t^2+1)(12t^2+3)\,dt$$

$$= \int_0^1 (12t^4 + 15t^2 + 3)\,dt$$

$$= \frac{12}{5} + \frac{15}{3} + 3 = \frac{12}{5} + 8 = \underline{\underline{\frac{52}{5}}}$$

Problem 6, continued.

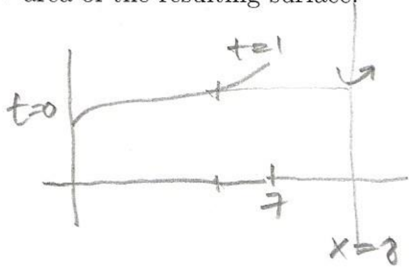
(c) Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating R around the x-axis.

$$V_1 = \pi \int_{t=0}^{t=1} y^2\,dx = \pi \int_0^1 (t^2+1)^2\,(12t^2+3)\,dt$$

(d) Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating R around the y-axis.

$$V_1 = 2\pi \int_{t=0}^{t=1} xy\,dx = 2\pi \int_0^1 (4t^3+3t)(t^2+1)(12t^2+3)\,dt$$

(e) Rotate the curve around the line $x = 8$. Set up, but do not evaluate, the integral for the surface area of the resulting surface.

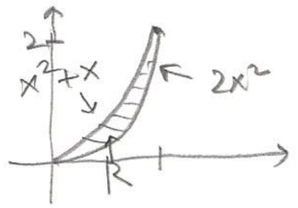


$$S.A. = 2\pi \int_{t=0}^{t=1} (8-x)\,ds$$

$$= 2\pi \int_0^1 (8 - 4t^3 - 3t^2) \cdot \sqrt{(12t^2+3)^2 + (2t)^2}\,dt$$

7. The region R lies in the first quadrant and is bounded by curves $y = x^2 + x$ and $y = 2x^2$. Compute the quantities below.

(a) Coordinates of the centroid of R. Express your answer in simple fractions.



$$\text{Area }(R) = \int_0^1 (x^2+x-2x^2)\,dx$$

$$= \int_0^1 (x - x^2)\,dx = \frac{1}{2} - \frac{1}{3} = \underline{\underline{\frac{1}{6}}}$$

$$M_y = \int_0^1 x\,(x^2+x-2x^2)\,dx = \int_0^1 (x^2-x^3)\,dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

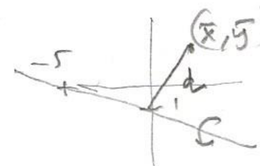
$$M_x = \frac{1}{2}\int_0^1 [(x^2+x)^2 - (2x^2)^2]\,dx = \frac{1}{2}\int_0^1 [-3x^4+2x^3+x^2]\,dx$$

$$= \frac{1}{2}\left(-\frac{3}{5} + \frac{1}{2} + \frac{1}{3}\right) = \frac{7}{60}$$

$$\bar{x} = \frac{M_y}{\text{Area}(R)} = \underline{\underline{\frac{1}{2}}}\,,\quad \underline{\underline{\bar{y}}} = \frac{M_x}{\text{Area}(R)} = \underline{\underline{\frac{7}{10}}}$$

$=$

(b) The volume of the solid obtained by rotating R around the line $2x + \overset{10}{\cancel{5}}y + 10 = 0$.



By Pappus

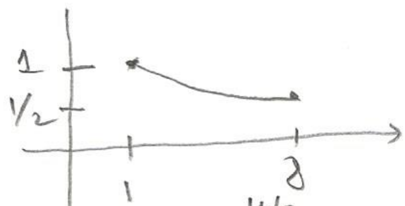
$$\text{Vol.} = 2\pi d \cdot \text{Area }(R)$$

$$= 2\pi\,\frac{\left(2\cdot\frac{1}{2} + 10\cdot\frac{7}{10} + 10\right)}{\sqrt{2^2+10^2}}\cdot\frac{1}{6}$$

$$= 2\pi\,\frac{18}{\sqrt{104}}\cdot\frac{1}{6} = \underline{\underline{\frac{3\pi}{\sqrt{26}}}}$$

8. Set up, but do not evaluate, the integrals for the arc length of the following curves. Also sketch roughly each of the curves (no concavity analysis necessary).

(a) $y = x^{-1/3}$, $1 \le x \le 8$, in Cartesian coordinates.

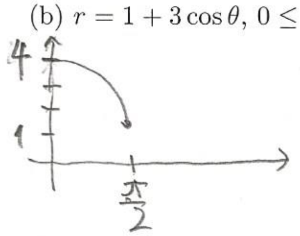


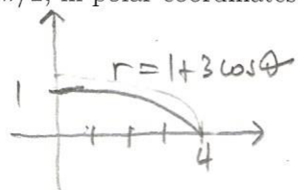
$$A.L. = \int_1^8 \sqrt{1 + \frac{1}{9}x^{-8/3}}\,dx$$

$$y' = -\frac{1}{3}x^{-4/3}$$

(b) $r = 1 + 3\cos\theta$, $0 \le \theta \le \pi/2$, in polar coordinates.

$$\frac{dr}{d\theta} = -3\sin\theta$$





$$A.L. = \int_0^{\pi/2} \sqrt{(1+3\cos\theta)^2 + 9\sin^2\theta}\,d\theta$$

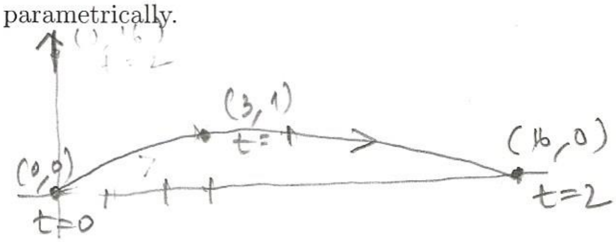
(c) $x = 2t^2 + t^3$, $y = 2t - t^2$, $0 \le t \le 2$, given parametrically.

$$\frac{dx}{dt} = 4t + 3t^2 \geq 0$$

$$\frac{dy}{dt} = 2 - 2t = 2(1-t)$$

$\frac{dy}{dt} > 0$ when $t < 1$

$\frac{dy}{dt} < 0$ when $t > 1$



$$A.L. = \int_0^2 \sqrt{(4t+3t^2)^2 + (2-2t)^2}\,dt$$