Exercises 11.1

Finding Cartesian from Parametric Equations

Exercises 1–18 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

1. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$
2. $x = -\sqrt{t}, \quad y = t, \quad t \geq 0$
3. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$
4. $x = 3 - 3t, \quad y = 2t, \quad 0 \leq t \leq 1$
5. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \leq t \leq \pi$
6. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \leq t \leq \pi$
7. $x = 4\cos t, \quad y = 2\sin t, \quad 0 \leq t \leq 2\pi$
8. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \leq t \leq 2\pi$
9. $x = \sin t, \quad y = \cos 2t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
10. $x = 1 + \sin t, \quad y = \cos t - 2, \quad 0 \leq t \leq \pi$
11. $x = t^2, \quad y = t^6 - 2t^4, \quad -\infty < t < \infty$
12. $x = \frac{t}{t - 1}, \quad y = \frac{t - 2}{t + 1}, \quad -1 < t < 1$
13. $x = t, \quad y = \sqrt{1 - t^2}, \quad -1 \leq t \leq 0$
14. $x = \sqrt{t + 1}, \quad y = \sqrt{t}, \quad t \geq 0$
15. $x = \sec^2 t - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$
16. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$
17. $x = -\cosh t, \quad y = \sinh t, \quad -\infty < t < \infty$
18. $x = 2\sinh t, \quad y = 2\cosh t, \quad -\infty < t < \infty$

Finding Parametric Equations

19. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$
    a. once clockwise.
    b. once counterclockwise.
    c. twice clockwise.
    d. twice counterclockwise.

    (There are many ways to do these, so your answers may not be the same as the ones in the back of the book.)

20. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $(x^2/a^2) + (y^2/b^2) = 1$
    a. once clockwise.
    b. once counterclockwise.
    c. twice clockwise.
    d. twice counterclockwise.

    (As in Exercise 19, there are many correct answers.)

In Exercises 21–26, find a parametrization for the curve.

21. the line segment with endpoints $(-1, -3)$ and $(4, 1)$
22. the line segment with endpoints $(-1, 3)$ and $(3, -2)$
23. the lower half of the parabola $x - 1 = y^2$
24. the left half of the parabola $y = x^2 + 2x$
25. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$
26. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$
27. Find parametric equations and a parameter interval for the motion of a particle starting at the point $(2, 0)$ and tracing the top half of the circle $x^2 + y^2 = 4$ four times.
28. Find parametric equations and a parameter interval for the motion of a particle that moves along the graph of $y = x^2$ in the following way: Beginning at $(0, 0)$ it moves to $(3, 9)$, and then travels back and forth from $(3, 9)$ to $(-3, 9)$ infinitely many times.
29. Find parametric equations for the semicircle

$$x^2 + y^2 = a^2, \quad y > 0,$$

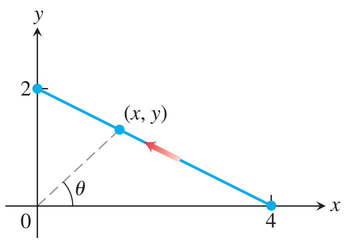
    using as parameter the slope $t = dy/dx$ of the tangent to the curve at (x, y).

30. Find parametric equations for the circle

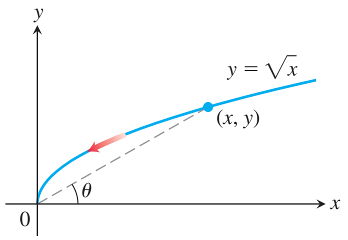
$$x^2 + y^2 = a^2,$$

    using as parameter the arc length s measured counterclockwise from the point $(a, 0)$ to the point (x, y).

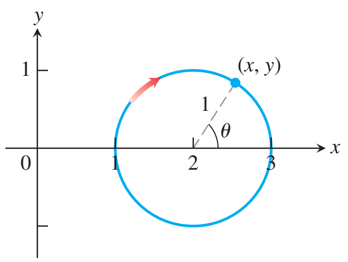
31. Find a parametrization for the line segment joining points $(0, 2)$ and $(4, 0)$ using the angle θ in the accompanying figure as the parameter.



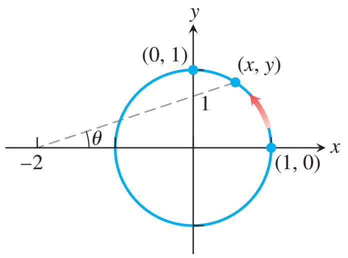
32. Find a parametrization for the curve $y = \sqrt{x}$ with terminal point $(0, 0)$ using the angle θ in the accompanying figure as the parameter.



33. Find a parametrization for the circle $(x - 2)^2 + y^2 = 1$ starting at $(1, 0)$ and moving clockwise once around the circle, using the central angle θ in the accompanying figure as the parameter.



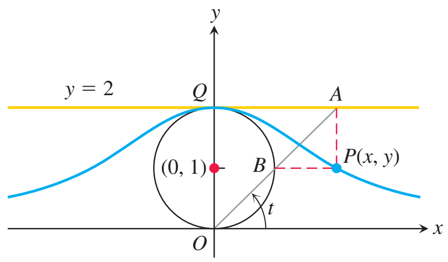
34. Find a parametrization for the circle $x^2 + y^2 = 1$ starting at $(1, 0)$ and moving counterclockwise to the terminal point $(0, 1)$, using the angle θ in the accompanying figure as the parameter.



35. The witch of Maria Agnesi The bell-shaped witch of Maria Agnesi can be constructed in the following way. Start with a circle of radius 1, centered at the point $(0, 1)$, as shown in the accompanying figure. Choose a point A on the line $y = 2$ and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The witch is the curve traced by P as A moves along the line $y = 2$. Find parametric equations and a parameter interval for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that segment O-A makes with the positive x-axis. The following equalities (which you may assume) will help.

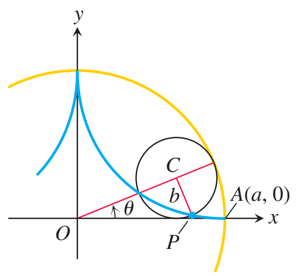
a. $x = AQ$ **b.** $y = 2 - AB \sin t$

c. $AB \cdot OA = (AQ)^2$

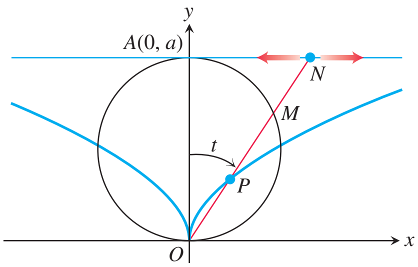


36. Hypocycloid When a circle rolls on the inside of a fixed circle, any point P on the circumference of the rolling circle describes a *hypocycloid*. Let the fixed circle be $x^2 + y^2 = a^2$, let the radius of the rolling circle be b, and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the hypocycloid, using as the parameter the angle θ from the positive x-axis to the line joining the circles' centers. In particular, if $b = a/4$, as in the accompanying figure, show that the hypocycloid is the astroid

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$



37. As the point N moves along the line $y = a$ in the accompanying figure, P moves in such a way that $OP = MN$. Find parametric equations for the coordinates of P as functions of the angle t that the line ON makes with the positive y-axis.



38. Trochoids A wheel of radius a rolls along a horizontal straight line without slipping. Find parametric equations for the curve traced out by a point P on a spoke of the wheel b units from its center. As parameter, use the angle θ through which the wheel turns. The curve is called a *trochoid*, which is a cycloid when $b = a$.

Distance Using Parametric Equations

39. Find the point on the parabola $x = t, y = t^2, -\infty < t < \infty$, closest to the point $(2, 1/2)$. (*Hint:* Minimize the square of the distance as a function of t.)

40. Find the point on the ellipse $x = 2\cos t, y = \sin t, 0 \le t \le 2\pi$ closest to the point $(3/4, 0)$. (*Hint:* Minimize the square of the distance as a function of t.)

T GRAPHER EXPLORATIONS

If you have a parametric equation grapher, graph the equations over the given intervals in Exercises 41–48.

41. Ellipse $x = 4\cos t$, $y = 2\sin t$, over

a. $0 \le t \le 2\pi$

b. $0 \le t \le \pi$

c. $-\pi/2 \le t \le \pi/2$.

42. Hyperbola branch $x = \sec t$ (enter as $1/\cos(t)$), $y = \tan t$ (enter as $\sin(t)/\cos(t)$), over

a. $-1.5 \le t \le 1.5$

b. $-0.5 \le t \le 0.5$

c. $-0.1 \le t \le 0.1$.

43. Parabola $x = 2t + 3$, $y = t^2 - 1$, $-2 \le t \le 2$

44. Cycloid $x = t - \sin t$, $y = 1 - \cos t$, over

a. $0 \le t \le 2\pi$

b. $0 \le t \le 4\pi$

c. $\pi \le t \le 3\pi$.

45. Deltoid

$$x = 2\cos t + \cos 2t, \quad y = 2\sin t - \sin 2t; \quad 0 \le t \le 2\pi$$

What happens if you replace 2 with −2 in the equations for x and y? Graph the new equations and find out.

46. A nice curve

$$x = 3\cos t + \cos 3t, \quad y = 3\sin t - \sin 3t; \quad 0 \le t \le 2\pi$$

What happens if you replace 3 with −3 in the equations for x and y? Graph the new equations and find out.