Math 21B, Winter 2022.
Feb. 4, 2022.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Throughout this problem, consider the integral $\int_2^4 \frac{1}{\ln x}\, dx$.

(a) Partition the interval $[2, 4]$ into 4 subintervals of equal length, and let the evaluation points c_k be the *right* endpoints. Write down (but do not evaluate) the resulting approximating (Riemann) sum to the integral.

$$\Delta X = \frac{1}{2}$$

$$\frac{1}{2}\left(\frac{1}{\ln 2.5} + \frac{1}{\ln 3} + \frac{1}{\ln 3.5} + \frac{1}{\ln 4}\right)$$

(b) Use the trapezoidal rule with 4 subintervals to write down an approximating expression for the above integral. (Do not evaluate this expression.)

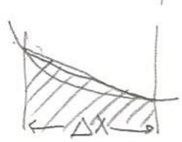
$$\frac{1}{4}\left(\frac{1}{\ln 2} + 2\cdot\frac{1}{\ln 2.5} + 2\cdot\frac{1}{\ln 3} + 2\cdot\frac{1}{\ln 3.5} + \frac{1}{\ln 4}\right)$$

(c) For each approximation (in (a) and in (b)), determine whether it overestimates or underestimates the integral.

$f(x) = \frac{1}{\ln x}$; $f'(x) = -\frac{1}{(\ln x)^2}\cdot\frac{1}{x} < 0$, so f is decreasing on $[2,4]$

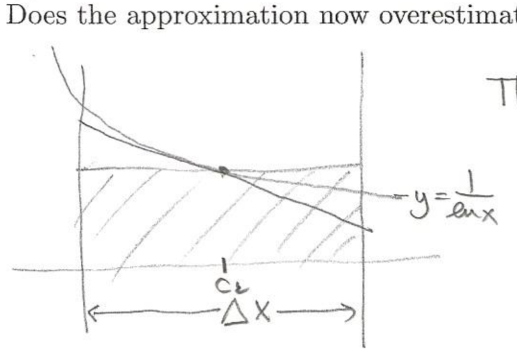
$f''(x) = \frac{2}{(\ln x)^3}\cdot\frac{1}{x} + \frac{1}{(\ln x)^2}\cdot\frac{1}{x^2} > 0$, so f is concave up on $[2,4]$

(a) underestimates



(b) overestimates

(d) Assume that now you use *midpoints* as evaluation points in (a), but keep everything else the same. Does the approximation now overestimate or underestimate the integral?



The area of the rectangle is the same as the area under the tangent at c_k; the tangent is below the graph; the approximation underestimates.

2. Throughout this problem, consider $F(x) = \int_1^x \frac{e^{t^2/8}}{t}\,dt$, and restrict x to $x \geq 1$.

(a) Determine the intervals on which $y = F(x)$ is increasing and those on which it is decreasing.

$F'(x) = \frac{e^{x^2/8}}{x} > 0$, The function is always increasing on $[1, \infty)$.

(b) Determine the intervals on which $y = F(x)$ is concave up and those on which it is concave down.

$$F''(x) = \frac{x \cdot \frac{x}{4} e^{x^2/8} - e^{x^2/8}}{x^2} = \frac{e^{x^2/8}(x^2-4)}{4x^2}$$

$$= \frac{e^{x^2/8}(x+2)(x-2)}{4x^2}$$

When $x > 2$, $F''(x) > 0$, F is concave up.

When $x < 2$, $F''(x) < 0$, F is concave down.

(c) Determine $\lim_{x \to 1+} \frac{F(x)}{\ln x}$. $= \lim_{x \to 1+} \frac{\frac{e^{x^2/8}}{x}}{1/x} = e^{1/8}$

$\left(\frac{0}{0}\right)$

3. Compute the following two antiderivatives.

(a) $\int \frac{2(\ln x)^3}{x}\,dx = 2\int u^3\,du = \frac{u^4}{2} + C$

$u = \ln x$

$dx = \frac{1}{x}dx$

$= \frac{(\ln x)^4}{2} + C$

(b) $\int \frac{x^5}{\sqrt{2x^3+5}}\,dx = \int \frac{x^3}{\sqrt{2x^3+5}} \cdot x^2\,dx$

$2x^3+5 = u, \quad x^3 = \frac{u-5}{2}$

$6x^2\,dx = du$

$x^2\,dx = \frac{1}{6}du$

$= \frac{1}{12}\int \frac{u-5}{\sqrt{u}}\,du = \frac{1}{12}\int \left(u^{1/2} - 5u^{-1/2}\right)du$

$= \frac{1}{12}\left[\frac{2}{3}u^{3/2} - 10\,u^{1/2}\right] + C$

$= \frac{1}{18}(2x^3+5)^{3/2} - \frac{5}{6}(2x^3+5)^{1/2} + C$

4. Compute the following two areas.
(a) Area under the graph of $y = \sqrt{1+3\sin x} \cdot \cos x$ on the interval $[0, \pi/2]$. Explain first why the function is defined and never negative on this interval. Give the result as a simple fraction.

$\sin x \geq 0$ on $[0, \pi/2]$, so $1 + 3\sin x \geq 1$ on $[0, \pi/2]$,
$\cos x \geq 0$ on $[0, \pi/2]$ so $\sqrt{1+3\sin x}$ is defined and positive

$$\text{Area} = \int_0^{\pi/2} \sqrt{1+3\sin x}\ \cos x \, dx$$

$1 + 3\sin x = u$ $\quad 3\cos x\, dx = du$, $\cos x\, dx = \frac{1}{3} du$

x	u
0	1
$\pi/2$	4

$$= \frac{1}{3}\int_1^4 u^{1/2}\, du = \frac{1}{3}\left[\frac{2}{3} u^{3/2} \Big|_{u=1}^{u=4} \right] = \frac{2}{9}\left(4^{3/2} - 1\right) = \underline{\underline{\frac{14}{9}}}$$

(b) Area of the bounded region enclosed by the graphs of $y = x^3 + 3x^2$ and of $y = x^3 + x^2 + 4x$. Give the result as a simple fraction.

Intersections: $\cancel{x^3} + 3x^2 = \cancel{x^3} + x^2 + 4x$
$2x^2 = 4x$, $x^2 = 2x$, $\underline{x = 0, 2}$

Which is larger in $(0,2)$: $x = 1$, $x^3 + 3x^2 = 4$
$x^3 + x^2 + 4x = 6$ larger

$$\text{Area} = \int_0^2 \left(\cancel{x^3} + x^2 + 4x - (\cancel{x^3} + 3x^2)\right) dx$$

$$= \int_0^2 (4x - 2x^2)\, dx = \left(2x^2 - \frac{2}{3}x^3\right)\Big|_{x=0}^{x=2}$$

$$= 8 - \frac{16}{3} = \underline{\underline{\frac{8}{3}}}$$