Math 21B, Fall 2023.
Oct. 25, 2023.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

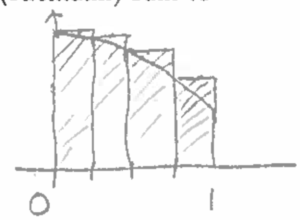
1	
2	
3	
4	
TOTAL	

x	$e^{-x^2/2}$
0	1
1/4	$e^{-1/32}$
1/2	$e^{-1/8}$
3/4	$e^{-9/32}$
1	$e^{-1/2}$

1. Throughout this problem, consider the integral $\int_0^1 e^{-x^2/2}\,dx$.

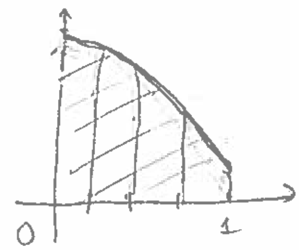
(a) Partition the interval $[0,1]$ into 4 subintervals of equal length, and let the evaluation points c_k be the *left* endpoints. Write down (but do not evaluate) the resulting approximating (Riemann) sum to the integral.

$$\frac{1}{4}\left[1 + e^{-1/32} + e^{-1/8} + e^{-9/32}\right]$$

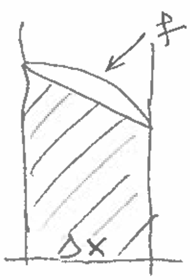


(b) Use the trapezoidal rule with 4 subintervals to write down an approximating expression for the above integral. (Do not evaluate this expression.)

$$\frac{1}{8}\left[1 + 2e^{-1/32} + 2e^{-1/8} + 2e^{-9/32} + 1\right]$$



(c) For each approximation (in (a) and (b)), determine whether it overestimates or underestimates the integral.



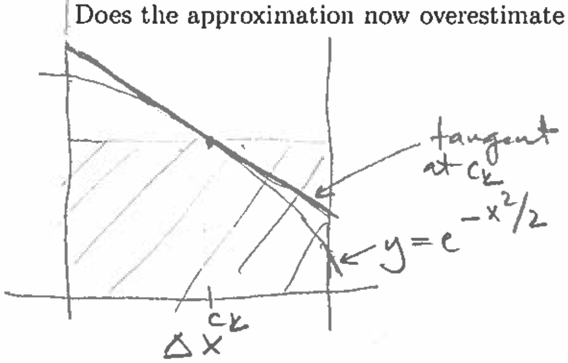
$$f(x) = e^{-x^2/2}$$

$$f'(x) = -x\,e^{-x^2/2} \leq 0$$ so f is decreasing, the approx in (a) <u>overestimates</u>

$$f''(x) = x^2 e^{-x^2/2} - 1\cdot e^{-x^2/2} = (x^2-1)\,e^{-x^2/2} \leq 0$$ for x in $[0,1]$

so the approx in (b) <u>underestimates</u>

(d) Assume that now you use *midpoints* as evaluation points in (a), but keep everything else the same. Does the approximation now overestimate or underestimate the integral?



The area of the rectangle is <u>the same</u> as the area under the tangent at c_k; the tangent is <u>above</u> the graph; the approx. <u>overestimates</u>.

2. Throughout this problem, consider $F(x) = \int_0^x \frac{8-t^3}{1+t^3}\,dt$, and restrict x to $x \geq 0$.

(a) Determine the intervals on which $y = F(x)$ increasing and those on which it is decreasing.

$$F'(x) = \frac{8-x^3}{1+x^3}$$

is > 0 when $x < 2$

where f is increasing

is < 0 when $x > 2$

where f is decreasing

(b) Determine the intervals on which $y = F(x)$ is concave up and those on which it is concave down.

$$F''(x) = \frac{-3x^2(1+\cancel{x^3}) - (8-\cancel{x^3}) \cdot 3x^2}{(1+x^3)^2} = \frac{-27x^2}{(1+x^3)^2} < 0$$

The function is always concave down.

(c) Determine $\lim_{x \to 0^+} \frac{F(x)}{x - x^3}$.

$\left(\frac{0}{0}\right)$

$$\overset{(L'H.)}{=} \lim_{x \to 0+} \frac{F'(x)}{1-3x^2} = \lim_{x \to 0+} \frac{\frac{8-x^3}{1+x^3}}{1-3x^2} = \underline{\underline{8}}$$

3. Compute the following two antiderivatives.

(a) $\displaystyle\int \frac{e^x}{e^x+3}\,dx \quad = \int \frac{du}{u} \quad = \ln u + C \quad = \ln(e^x+3) + C$

$e^x + 3 = u$

$e^x dx = du$

(b) $\displaystyle\int 8\sqrt{2x^2+5}\cdot x^3\,dx \quad = \int 8\sqrt{2x^2+5}\cdot x^2 \cdot x\,dx$

$2x^2+5 = u \qquad x^2 = \frac{u-5}{2}$

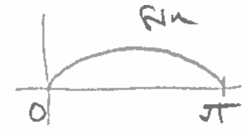
$4x\,dx = du$

$= \int \cancel{8}\sqrt{u}\cdot(u-5)\cdot \cancel{\frac{1}{2}}\cdot\cancel{\frac{1}{4}}\,du$

$= \int \left(u^{3/2} - 5u^{1/2}\right) du$

$= \frac{2}{5}u^{5/2} - 5\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(2x^2+5)^{5/2} - \frac{10}{3}(2x^2+5)^{3/2} + C$



4. Compute the following two areas.
(a) Area under the graph of $y = (\cos(4x))^2 \cdot \sin(4x)$ on the interval $[0, \pi/4]$. Explain first why the function is never negative on this interval. Give the result as a simple fraction.

$(\cos(4x))^2 \geq 0$ for all x; $\sin(4x) \geq 0$ on $[0, \frac{\pi}{4}]$
(as $4x$ in $[0, \pi]$, where sin is ≥ 0)

$$\int_0^{\pi/4} (\cos 4x)^2 \sin(4x)\, dx$$

$\cos(4x) = t \qquad -4 \sin x\, dx = dt$

x	t
0	1
$\pi/4$	-1

$$= \int_1^{-1} t^2 \left(-\frac{1}{4}\right) dt = \frac{1}{4} \int_{-1}^{1} t^2 dt = \frac{1}{2} \int_0^1 t^2 dt$$

$$= \frac{1}{2} \left.\frac{t^3}{3}\right|_0^1 = \underline{\underline{\frac{1}{6}}}$$

(b) Area of the bounded region enclosed by the graphs of $y = x^5$ and of $y = x^5 + x^2 - 2x$. Give the result as a simple fraction.

Intersection: $x^5 = x^5 + x^2 - 2x \qquad x^2 - 2x = 0$
$x(x-2) = 0$, $x = 0, 2$

At 1: $y = x^5$ is 1
$y = x^5 + x^2 - 2x$ is 0, so x^5 is larger for x in $(0, 2)$

$$\text{Area} = \int_0^2 \left[x^5 - (x^5 + x^2 - 2x)\right] dx$$

$$= \int_0^2 (2x - x^2)\, dx = x^2 - \left.\frac{x^3}{3}\right|_0^2$$

$$= 4 - \frac{8}{3} = \underline{\underline{\frac{4}{3}}}$$