Math 21B, Fall 2023.
Nov. 17, 2023.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$
$$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$
$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$
$$\sin^2 A = \frac{1}{2}(1 - \cos(2A)), \quad \cos^2 A = \frac{1}{2}(1 + \cos(2A))$$

1. Compute the following two indefinite integrals.

(a) $\displaystyle\int \frac{x+1}{(x-1)^2(x+2)}\,dx$

$$\frac{x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$* (x-1)^2, \ x \to 1 : \quad B = \frac{2}{3}$

$* (x+2), \ x \to -2 : \quad C = -\frac{1}{9}$

$* x, \ x \to \infty ; A + C = 0, \ A = \frac{1}{9}$

$$= \frac{1}{9}\int \frac{dx}{x-1} + \frac{2}{3}\int \frac{dx}{(x-1)^2} - \frac{1}{9}\int \frac{dx}{x+2}$$

$$= \frac{1}{9}\ln|x-1| - \frac{2}{3}\cdot\frac{1}{x-1} - \frac{1}{9}\ln|x+2| + C$$

(b) $\displaystyle\int \sin^2 x \cos(3x)\,dx$

$$= \int \frac{1}{2}(1-\cos 2x)\cos 3x \, dx$$

$$= \frac{1}{2}\int \cos 3x \, dx - \frac{1}{2}\int \cos 2x \cos 3x \, dx$$

$$= \frac{1}{2}\cdot\frac{1}{3}\sin 3x - \frac{1}{4}\int [\cos x + \cos 5x] dx$$

$$= \frac{1}{6}\sin 3x - \frac{1}{4}\sin x - \frac{1}{20}\sin 5x + C$$

2. Compute the following two definite integrals.

(a) $\int_0^2 \ln(x+1)\,dx$

$u = \ln(x+1) \qquad du = \frac{1}{x+1}\,dx$

$dv = dx \qquad v = x$

$\frac{x}{x+1} = 1 - \frac{1}{x+1}$

$= x \ln(x+1)\Big|_0^2 - \int_0^2 \frac{x}{x+1}\,dx$

$= 2\ln 3 - \left[x - \ln(x+1)\right]_0^2$

$= 2\ln 3 - 2 + \ln 3 = \underline{\underline{3\ln 3 - 2}}$

(b) $\int_{-1}^2 \frac{1}{x^2+2x+10}\,dx$

$x^2+2x+10 = (x+1)^2+9 = 9(u^2+1)$

$x+1 = 3u$

$dx = 3\,du$

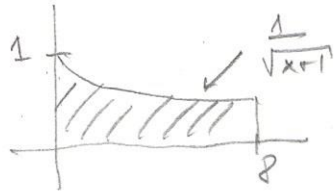
x	u
-1	0
2	1

$= \int_0^1 \frac{1}{9(u^2+1)} \cdot 3\,du$

$= \frac{1}{3}\int_0^1 \frac{1}{u^2+1}\,du = \frac{1}{3}\arctan u\Big|_0^1 = \frac{1}{3}\cdot\frac{\pi}{4} = \underline{\underline{\frac{\pi}{12}}}$

3. The region R lies between $x = 0$ and $x = 8$ and is bounded by the graphs of $y = \dfrac{1}{\sqrt{x+1}}$ and $y = 0$.

(a) Rotate R around the x-axis and compute the volume of the resulting solid.



Washer method:

$$\text{Vol.} = \pi \int_0^8 \left(\frac{1}{\sqrt{x+1}}\right)^2 dx$$

$$= \pi \int_0^8 \frac{1}{x+1}\, dx$$

$$= \pi \ln(x+1) \Big|_0^8$$

$$= \pi \ln 9 = \underline{\underline{3\pi \ln 3}}$$

(b) Rotate R around the line $x = -1$ and compute the volume of the resulting solid.

Shell method

$$\text{Vol.} = 2\pi \int_0^8 \overbrace{(x+1)}^{\text{radius}}\ \overbrace{\frac{1}{\sqrt{x+1}}}^{\text{height}}\, dx$$

$$= 2\pi \int_0^8 \sqrt{x+1}\, dx$$

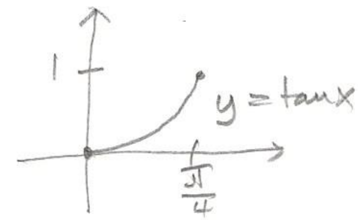
$$= 2\pi \left.\frac{(x+1)^{3/2}}{3/2}\right|_0^8$$

$$= \frac{4\pi}{3} \underbrace{\left(9^{3/2} - 1\right)}_{=26} = \underline{\underline{\frac{104\pi}{3}}}$$

4. Consider the curve given as the graph of the function $y = \tan x$, for $0 \le x \le \pi/4$. Write, *but do not compute* the integrals for quantities specified below.

(a) The arc length of this curve.

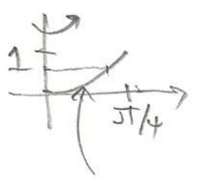
$$y' = \frac{1}{\cos^2 x}$$



$$\text{A.L.} = \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \frac{1}{\cos^4 x}}\, dx$$

(b) The surface area of the surface obtained by revolution of this curve around the y-axis.

$$\text{Area} = 2\pi \int_0^{\pi/4} x\, ds = 2\pi \int_0^{\pi/4} x \sqrt{1 + \frac{1}{\cos^4 x}}\, dx$$



or

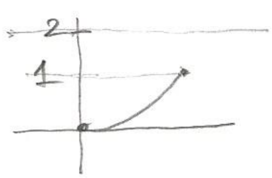
$$= 2\pi \int_0^1 \arctan y \cdot \sqrt{1 + \left(\frac{1}{1+y^2}\right)^2}\, dy$$

$x = \arctan y$

$\frac{dx}{dy} = \frac{1}{1+y^2}$

(Both are correct answers!)

(c) The surface area of the surface obtained by revolution of this curve around the line $y = 2$. (Explain why this line does not intersect the curve!)



$y = \tan x$ increases from 0 to 1 on the interval $[0, \pi/4]$, so it stays below the line $y = 2$

$$\text{Area} = 2\pi \int_0^{\pi/4} \underbrace{(2 - \tan x)}_{\text{radius}} \underbrace{\sqrt{1 + \frac{1}{\cos^4 x}}\, dx}_{ds}$$