# Martingales: definition and basic properties

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# Martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$$

be an increasing sequence of $\sigma$-algebras, a *filtration*, with $\mathcal{F}_\infty = \sigma(\cup_{n=1}^{\infty} \mathcal{F}_n)$.
$\mathcal{F}_n$ is "the information available at time $n$."

Let $X_n$ be a sequence of random variables. We say that $X_n$ is *adapted to* $\mathcal{F}_n$ if $X_n$ is $\mathcal{F}_n$-measurable.

# Martingales

## Definition

Assume that $X_n$, $n = 0, 1, 2, \ldots$, is a sequence of r.v.'s and $\mathcal{F}_n$ is a filtration. Then $X_n$ is a *martingale* (w.r.t. $\mathcal{F}_n$) if for all $n \geq 0$,

(i) $E|X_n| < \infty$;

(ii) $X_n$ is adapted to $\mathcal{F}_n$; and

(iii) $E[X_{n+1} \mid \mathcal{F}_n] = X_n$ a.s.

A *supermatingale* has $\leq$ in (iii) and a *submatingale* has $\geq$ in (iii).

# Martingales: observations

(1) Observe that $EX_{n+1} \geq EX_n$ for a submartingale.

(2) The filtration

$$\mathcal{F}_n^{\mathrm{nat}} = \sigma\{X_0, \ldots, X_n\}$$

is the *natural* filtration. Note that $\mathcal{F}_n^{\mathrm{nat}} \subset \mathcal{F}_n$, and for a submartingale,

$$E[X_{n+1}|\mathcal{F}_n^{\mathrm{nat}}] = E[E[X_{n+1} \mid \mathcal{F}_n] \mid \mathcal{F}_n^{\mathrm{nat}}] \geq E[X_n \mid \mathcal{F}_n^{\mathrm{nat}}] = X_n$$

so we keep (iii) if we replace the original filtration by $\mathcal{F}_n^{\mathrm{nat}}$. This is the minimal filtration for (i)–(iii). When unstated, we will assume that this is the filtration used.

# Martingales: examples

**Example.** Sum of mean-zero independent r.v.'s: if $\xi_1, \xi_2, \ldots$ are independent and $E\xi_i = 0$, then the sequence $S_n$ given by $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$ for $n \geq 1$, is a martingale. (When $E\xi_i \geq 0$ it is a submartingale and when $E\xi_i \leq 0$ it is a supermartingale.)

# Martingales: examples

**Example.** Product of mean-one independent r.v.'s: if $\xi_1, \xi_2, \ldots$ are independent and $E\xi_i = 1$, then the sequence $R_n$ given by $R_0 = 1$, $R_n = \xi_1 \cdots \xi_n$ for $n \geq 1$, is a martingale. (If $\xi_i \geq 0$, then when $E\xi_i \geq 1$ it is a submartingale and when $E\xi_i \leq 1$ it is a supermartingale.)

# Martingales: examples

**Example.** Levy or Doob martingale: if $E|X| < \infty$ and $\mathcal{F}_n$ is any filtration, then $X_n = E[X \mid \mathcal{F}_n]$ is a martingale.

This follows from the tower property.

# Martingales: examples

**Example.** Galton-Watson (or branching) process: Start with one subject in generation 0. Every subject in every generation produces an independent number of offspring in the next generation. This number has *offspring distribution* given by $p_k = P(\text{no. of children} = k)$, $k = 0, 1, 2, \ldots$

Let $\mu = \sum_{k=0}^{\infty} kp_k$.

# Martingales: examples

Rigorous setting: let the *potential* numbers of offspring

$$\xi_1^1, \xi_2^1, \xi_3^1, \dots$$
$$\xi_1^2, \xi_2^2, \xi_3^2, \dots$$
$$\dots$$

be i.i.d. with offspring distribution. Note that $E\xi_i^m = \mu$.

Let $\mathcal{F}_n = \sigma\{\xi_i^m : 1 \leq m \leq n\}$ (with $\mathcal{F}_0$ trivial). Let $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} 0 & \text{if } Z_n = 0 \\ \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1} & \text{if } Z_n \geq 1 \end{cases}$$

# Martingales: examples

## Proposition

*Assume $\mu < \infty$. Then $Z_n/\mu^n$ is a martingale.*

## Proof.

Observe that $Z_n$ is $\mathcal{F}_n$-measurable. Further,

$$
\begin{aligned}
E[Z_{n+1} \mid \mathcal{F}_n] &= \sum_{k=1}^{\infty} E[Z_{n+1} 1_{\{Z_n = k\}} \mid \mathcal{F}_n] \\
&= \sum_{k=1}^{\infty} E[(\xi_1^{n+1} + \ldots + \xi_k^{n+1}) 1_{\{Z_n = k\}} \mid \mathcal{F}_n] \\
&= \sum_{k=1}^{\infty} 1_{\{Z_n = k\}} k\mu = \mu Z_n
\end{aligned}
$$

So, $EZ_n = \mu^n < \infty$, and $E[Z_{n+1}/\mu^{n+1} \mid \mathcal{F}_n] = Z_n/\mu^n$. □

# Martingales: observations

(3) If $X_n$ is a submartingale, then $E[X_n \mid \mathcal{F}_m] \geq X_m$ for all $m \leq n$. (with $\leq$ for supermartingales, $=$ for martingales).

**Proof.**

We have
$E[X_{m+2} \mid \mathcal{F}_m] = E[E[X_{m+2} \mid \mathcal{F}_{m+1}] \mid \mathcal{F}_m] \geq E[X_{m+1} \mid \mathcal{F}_m] \geq X_m$,
etc. □

# Martingales: observations

(4) If $X_n$ is a martingale, and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex with $E|\varphi(X_n)| < \infty$ for all $n$, then $\varphi(X_n)$ is a submartingale.

**Proof.**

By conditional Jensen, we have $E[\varphi(X_{n+1}) \mid \mathcal{F}_n] \geq \varphi(E[X_{n+1} \mid \mathcal{F}_n]) = \varphi(X_n)$. □

If $\varphi$ is concave and $X_n$ is a martingale, then $\varphi(X_n)$ is a supermartingale.

If $\varphi$ is convex and nondecreasing and $X_n$ is a submartingale, then $\varphi(X_n)$ is a submartingale. (The supermartingale version: concave and nondecrasing.)

# Martingales: observations

Important special cases:

If $X_n$ is a martingale and $EX_n^2 < \infty$, then $X_n^2$ is a submartingale.

If $X_n$ is a submartingale then $X_n^+$ is a submartingale.

If $X_n$ is a supermartingale then
$X_n \wedge a = \min\{X_n, a\} = a - (a - X_n)_+$ is a supermartingale.

# Martingale transform

Let $\mathcal{F}_n$, $n \geq 0$ be a filtration. A sequence $H_n$, $n \geq 1$ of r.v.'s is *predictable* if $H_n$ is $\mathcal{F}_{n-1}$-measurable, for all $n \geq 1$. (There is no $H_0$.)

The basic interpretation is the amount of money a gambler bets at the $n$th game.

## Definition (Martingale transform)

Let

$$
\begin{aligned}
(H \bullet X)_0 &= 0 \\
(H \bullet X)_n &= \sum_{m=1}^{n} H_m (X_m - X_{m-1})
\end{aligned}
$$

# Martingale transform

**Theorem**

*Assume $X_n$ is a supermartingale. If $H_n \geq 0$ and each $H_n$ is bounded (i.e., $H_n \leq c_n \in \mathbb{R}$, for all n), then $(H \bullet X)_n$ is a supermartingale. Same for submartingales. Same also for martingales, except that we do not need $H_n \geq 0$ in this case.*

**Proof.**

Boundedness is needed for $E|(H \bullet X)_n| < \infty$. Moreover, for supermartingales,

$$E[(H\bullet X)_{n+1} \mid \mathcal{F}_n] = (H\bullet X)_n + H_{n+1} E[(X_{n+1} - X_n) \mid \mathcal{F}_n] \leq (H\bullet X)_n.$$

□

# Martingale transform

**Example.** *The* martingale. Assume that $X_n = \xi_1 + \cdots + \xi_n$, where $\xi_i$ are i.i.d. with $P(\xi_i = 1) = p \leq 1/2$ and $P(\xi_i = -1) = 1 - p$.
Then $X_n$ is a supermartingale and $X_m - X_{m-1} = \xi_m$. Let $H_n$ be the amount of money the gambler bets (on $\xi_n = 1$) at time $n$, so that $(H \bullet X)_n$ are the total winnings through time $n$.
Using "double when you lose" strategy, define

$$H_1 = 1$$
$$H_n = \begin{cases} 2H_{n-1} & \text{if } \xi_{n-1} = -1 \\ 1 & \text{otherwise} \end{cases}$$

Let $N$ be the time of the first win (when $\xi_n = 1$ for the first time). Then $(H \bullet X)_N = 1$. But $(H \bullet X)_n$ is 0 at $n = 0$ and is a supermartingale. Contradiction?

# Martingale transform

No! The time $N$ is *random*!

# Stopping time

**Definition**

A random variable $N \in \{0, 1, 2, \ldots, \infty\}$ is a *stopping time* w.r.t. filtration $\mathcal{F}_n$ if $\{N = n\} \in \mathcal{F}_n$ for every $n = 0, 1, 2, \ldots$.

Note that $N = \infty$ is a possible value. Also note that we could equivalently require that $\{N \leq n\} \in \mathcal{F}_n$ for every finite $n$.

In the gambling context, the gambler's decision when to stop may be random, but must be based on the information available at time $n$.

# Stopping time

**Example.** Pick a $p \in (0, 1)$. Let $\xi_i$ be i.i.d., $P(\xi_i = 1) = p$, $P(\xi_i = -1) = 1 - p$, and $S_n = \sum_{i=1}^{n} \xi_i$, a *one-dimensional simple random walk*. Take $\mathcal{F}_n = \sigma\{\xi_1, \dots, \xi_n\}$ (with $\mathcal{F}_0 = \{0, \Omega\}$).

Then $T = \inf\{k : S_k \geq 10\}$ is a stopping time, as

$$\{T \leq n\} = \cup_{0 \leq k \leq n}\{S_k \geq 10\}.$$

Note that it is possible that $T = \infty$, which happens with positive probability when $p < 1/2$ (as we will see).

On the other hand, $T' = \sup\{k \leq 20 : S_k \geq 0\}$ is not a stopping time; for example

$$\{T' = 0\} = \cap_{k=1}^{20}\{S_k < 0\} \notin \mathcal{F}_0.$$

# Stopping time

The main example of a stopping time is a *hitting time*. For $A \in \mathcal{B}(\mathbb{R})$, and $X_n$ a process adapted to filtration $\mathcal{F}_n$, we define the hitting time

$$\tau_A = \inf\{n \geq 0 : X_n \in A\}.$$

As

$$\{\tau_A \leq n\} = \cup_{k \leq n}\{X_k \in A\},$$

this is a stopping time.

**Proposition**

*If $S, T$ are stopping times, then so are $S \wedge T = \min\{S, T\}$ and $S \vee T = \max\{S, T\}$.*

Proof for HW.

# Stopping time

Assume that $X_n$ is adapted to a filtration $\mathcal{F}_n$, and $N$ is a stopping time for the same filtration.
Let $H_n = 1_{\{N \geq n\}} = 1_{\{N \leq n-1\}^c}$. This is a nonnegative predictable process, and

$$(H \bullet X)_n = \sum_{k=1}^{n} 1_{\{N \geq k\}} (X_k - X_{k-1}) = \sum_{k=1}^{N \wedge n} (X_k - X_{k-1}) = X_{N \wedge n} - X_0.$$

The process $X_{N \wedge n}$ is called the *stopped process*, and is clearly also adapted.

# Stopping time

**Corollary**

*If $X_n$ is a supermartingale and $N$ is a stopping time (with same filtration), then the stopped process is also a supermartingale. Therefore, $EX_{N \wedge n} \leq EX_0$. (Same for a submartingale and a martingale.)*

**Proof.**

This follows from the martingale transform theorem. □

# Stopping time

Assume that $X_n$ is a supermartingale and $\tau$ is a *finite* stopping time, i.e., $P(\tau < \infty) = 1$. Assume that either: (a) $\tau$ is bounded, i.e., $P(\tau \le a) = 1$ for some integer $a$; or (b) the stopped process is uniformly bounded, i.e., $\sup_n |X_{\tau \wedge n}| \le c$ for some $c \in \mathbb{R}$. Then $EX_\tau = EX_0$.

This is a special case of the *optional stopping theorem*, which we will prove is some generality later. Note that (a) and (b) both fail for the martingale.

**Proof.**

We know that $E(X_{n \wedge \tau}) \le EX_0$, and we may send $n \to \infty$, using DCT in the case (b). □

# Stopping time

**Example.** Let $S_n$ be a simple symmetric one-dimensional random walk, that is, $p = 1/2$. (In particular, $S_0 = 0$.) Fix $a, b \geq 0$ and let $T = \inf\{n : S_n = -a \text{ or } S_n = b\}$. Then $S_n$ and $S_n^2 - n$ are martingales with $\mathcal{F}_n = \sigma\{\xi_1, \ldots, \xi_n\}$. To check,

$$\begin{aligned} & E[S_{n+1}^2 - (n+1) \mid \mathcal{F}_n] \\ & = E[S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2 - n - 1 \mid \mathcal{F}_n] = S_n^2 - n. \end{aligned}$$

Then $P(T < \infty) = 1$ as $P(T > n) \leq (1 - 2^{-a-b})^{\lfloor n/(a+b) \rfloor}$ and $|S_{n \wedge T}| \leq \max\{a, b\}$.
So, $ES_T = 0$, that is, $0 = -aP(S_T = -a) + bP(S_T = b)$, which implies that

$$P(S_T = -a) = \frac{b}{a+b}, P(S_T = b) = \frac{a}{a+b}$$

# Stopping time

We also know that for all $n$,

$$E(S_{T\wedge n}^2) = E(T \wedge n).$$

Send $n \to \infty$, using DCT on LHS and MCT on RHS, to get

$$ET = ES_T^2 = a^2 P(S_T = -a) + b^2 P(S_T = b) = ab.$$

Let $S$ is the hitting time of 1, assume $b = 1$, and call previous time $T_a$. Then

$$P(S = \infty) = P(\cap_{a=1}^{\infty} \{S_{T_a} = -a\}) = \lim_{a\to\infty} P(S_{T_a} = -a) = 0,$$

and $T_a \uparrow S$ as $a \to \infty$, so $ES = \lim ET_a = \infty$. More on this later.