Math 236A, Fall 2012.

Homework Assignment 3

Due: Oct. 26, 2012

1. Let τ_a be the first time a standard Brownian motion in one dimension hits $a > 0$.
(a) Compute the density of τ_a.
(b) Either from (a) or (much more easily) by using an appropriate martingale, compute $E(\exp(-\lambda\tau_a))$, for any $\lambda > 0$.
(c) Let $b > 0$ and let τ_{-b} be the first time the Brownian motion hits $-b$. Show that

$$E(\exp(-\lambda\tau_{-b})1_{\{\tau_a<\tau_{-b}\}}) = E(\exp(-\lambda\tau_a)1_{\{\tau_a<\tau_{-b}\}}) \cdot E(\exp(-\lambda\tau_{a+b})).$$

(d) Let $\tau = \tau_a \wedge \tau_{-a}$. Compute $E(\exp(-\lambda\tau))$.

2. Let B be the Brownian motion in two dimensions, started at $(0, a)$, $a > 0$. Let now τ be the first time B hits the line αx. Also, let X be the x-coordinate of the point $B(\tau)$.
(a) Determine the density of X when $\alpha = 0$. (*Hint*. Condition on the value of the stopping time τ_a from problem 1(a).)
(b) Now determine the density of X when $\alpha \neq 0$. (*Hint*. Use orthogonal invariance.)