Math 236A, Fall 2012.

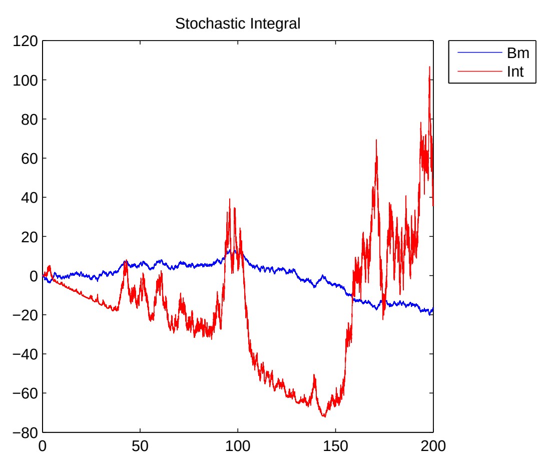
Homework Assignment 4

Due: Nov. 9, 2012

1. Show that the following processes are martingales, and compute their variances.
(a) $X_t = \int_0^t |B_s|\, dB_s$
(b) $X_t = \int_0^t M_s\, dB_s$, where $M_s = \max\{B_u : 0 \le u \le s\}$.
(c) $X_t = \int_0^t B_{s/2}\, dB_s$,
(d) $X_t = \int_0^t I_s\, dB_s$, where $I_s = \int_0^s B_u^2\, du$.
(e) $X_t = \int_0^t B_s \bar{B}_s\, dB_s$, where $\bar{B}_s$ is a Brownian motion independent of B_s.
(f) $X_t = \int_0^t B_{\xi s}\, dB_s$, where ξ is a uniform random variable on $[0, 1]$, independent of B_s.

2. Consider $\int_0^t B_{s+1}\, dB_s$, $0 \le t \le 1$.
(a) This is not an Itô integral. Why?
(b) Still, one may hope that, as $n \to \infty$, the limit of the approximating sums $\sum_{i=0}^{n-1} B_{t_i+1}(B_{t_{i+1}} - B_{t_i})$ exists in an appropriate sense. Show that the limit indeed exists (state in what sense), express it with genuine Itô integrals, and show that it results in a continuous process which is *not* a martingale.

3. The figure depicts a realization of $B_{k/n}$ (blue) and $\sum_{i=0}^{k-1} B_{i/n}(B_{(i+1)/n} - B_{i/n})$ (red) for a large n, and $0 \le k/n \le 200$ (plotted against k/n and interpolated). Explain the presence of almost linear segments in the red curve and its much larger oscillations.



4. Compute the following stochastic integrals. The answers may contain Riemann integrals.
(a) $\int_0^t e^s\, dB_s$
(b) $\int_0^t B_s^2\, dB_s$
(c) $\int_0^t e^{B_s}\, dB_s$.
(d) $\int_0^t sB_s\, dB_s$.

5. Fix $a > 0$ and let τ be the first time that Brownian motion B_t hits either a or $-a$. Compute, with careful proof:

$$E \int_0^\tau B_s^2 \, ds.$$

(*Hint*: Itô's formula.)