Math 236A, Fall 2012.

Homework Assignment 5

Due: Nov. 16, 2012

1.
(a) Assume that $f = f(t,x)$ is a $\mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ such that $\partial_t f + \frac{1}{2}\partial_{xx} f = 0$. Show that $f(t, B_t)$ is a local martingale.
(b) For f as in (a), let $g(t,x) = \int_0^x f(t,z)\,dz - \frac{1}{2}\int_0^t \partial_x f(s,0)\,ds$. Show that $g(t,B_t)$ is also a local martingale.
(c) Let h_n be the Hermite polynomial of degree n. (The first four are 1, x, x^2-1, x^3-3x.) Let $f_n(t,x) = t^{n/2}h_n(x/\sqrt{t})$ for $n \geq 0$. Show that $f_n(t,B_t)$ is a martingale for every n.
(d) Define the processes X_t^n recursively by $X_t^0 \equiv 1$ and $X_t^{n+1} = \int_0^t X_s^n\,ds$. Find the connection between X_t^n and the martingales from (c).

2. Assume that B_t^1 and B_t^2 are independent Brownian motions. Consider $X_t = \int_0^t B_s^1\,dB_s^1$ and $X_t = \int_0^t B_s^2\,dB_s^1$. Then the two bracket processes $\langle X \rangle_t$ and $\langle Y \rangle_t$ are equal in distribution, but it is not even true that X_t and Y_t are equal in distribution for a fixed $t > 0$. Prove this.

3. What should be the quadratic variation process for a process given by $dX_t = a\,dt + b_1\,dB_t^1 + b_2\,dB_t^2$? Give the reasoning for your formula although a complete proof is not necessary. In particular, compute the quadratic variation process in the case $dX_t = B_t^1 B_t^2\,dB_t^1 + |B_t^1|\,dB_t^2$.