Math 236A, Fall 2012.

Homework Assignment 7

Due: Dec. 10, 2012, noon. Please leave the assignment in my mailbox or slide it under the door of my office. *This assignment covers the entire course and you are required to make a credible attempt at every problem.*

1. Let τ be a stopping time, with $0 \le \tau \le 1$, for the Brownian motion B_t and $I = \int_0^1 B_t\,dt$.
(a) Show (carefully!) that $E(B_t B_\tau) = E(\tau \wedge t)$ for each $t \in [0,1]$.
(b) Find a formula for

$$E[(I - B_\tau)^2],$$

in terms of $E\tau$, $E\tau^2$.
(c) Find a stopping time $\tau \in [0,1]$ that minimizes the expression in (a).

2. Assume that $h : [0,1] \to \mathbb{R}$ is an absolutely continuous (deterministic) function with L^2 derivative.
(a) Show that for any $\epsilon > 0$, $P(|B_t| < \epsilon$ for all $t \in [0,1]) > 0$.
(b) Assume that $h : [0,1] \to \mathbb{R}$ is an absolutely continuous (deterministic) function with an L^2 derivative, and with $h(0) = 0$. Show that for any $\epsilon > 0$, $P(|B_t - h(t)| < \epsilon$ for all $t \in [0,1]) > 0$.
(c) Let h be as in (b). Show that, with probability 1, there exists an integer n so that $|B_n| < \epsilon$ and $|B_{t+n} - B_n - h| < \epsilon$ for all $t \in [0,1]$.
(d) "A two-dimensional Brownian motion will eventually write your name near the origin." Formulate this precisely (with all necessary assumptions) and explain how it follows from (c).

3. Consider the two processes

$$X_t = \int_0^t \cos(s)\,dB_s^1 \qquad \text{and} \qquad Y_t = \int_0^t \sin(s)\,dB_s^2.$$

(a) Assume first that $B_t^1 = B_t^2 = B_t$ is the same Brownian motion. For which times t are the random variables X_t and Y_t independent?
(b) Assume now that B_t^1 and B_t^2 are independent Brownian motions. Compute the bracket $\langle Z \rangle_t$ of the process $Z_t = X_t + Y_t$. What can you say about the distribution of the process Z_t?

4. Find the unique solution to the SDE

$$dX_t = a\,dt + \sigma X_t\,dB_t$$

with $X_0 = 0$. Here, a and σ are fixed constants.

5. Consider logistic growth with multiplicative noise, that is, the SDE

$$dX_t = (\lambda X_t - X_t^2)\,dt + \sigma X_t\,dB_t.$$

Here, $\lambda > 0$ and $\sigma > 0$ are fixed constants. Find the unique solution for any deterministic $X_0 = x_0 > 0$ explicitly, by deriving the equation for $Y_t = 1/X_t$. What can you say about the behavior of X_t as $t \to \infty$?

6. In the Black-Scholes model, consider the *All or Nothing* option that pays one unit of cash at time T provided the stock exceeds the value K some time during the interval $[0, T]$; otherwise it pays 0. That is, its value at time T is

$$X = 1_{\{\max_{t \in [0,T]} S_t \geq K\}}.$$

Find the arbitrage price of X at time 0.