ON FINITENESS OF THE NUMBER OF BOUNDARY SLOPES OF IMMERSED SURFACES IN 3-MANIFOLDS

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Abstract. For any hyperbolic 3-manifold M with totally geodesic boundary, there are finitely many boundary slopes for essential immersed surfaces of a given genus. There is a uniform bound for the number of such boundary slopes if the genus of ∂M or the volume of M is bounded above. When the volume is bounded above, then area of ∂M is bounded above and the length of closed geodesic on ∂M is bounded below.

We say that a proper immersion of a surface F into M is an *essential surface* if it is incompressible and ∂-incompressible, meaning that the immersion induces an injection of the fundamental group and relative fundamental group. Let c be an essential simple loop on the boundary ∂M of a compact 3-manifold M. If there is a proper immersion of an essential surface F into M such that each component of ∂F is homotopic to a multiple of c, we call c a *boundary slope* of F.

We are interested in the following two questions:

Questions.

(1) Given a compact 3-manifold M and a genus g, are there finitely many boundary slopes for immersed essential surfaces with genus at most g?

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(2) Under what conditions is there a bound for the number of boundary slopes in (1) which is independent of the 3-manifold?

Many results in these directions have been obtained for various classes of 3-manifolds:

(1) If ∂M is a torus and the surfaces are embedded, Hatcher [H] showed that there are only finitely many boundary slopes, without any genus restriction.

(2) When the surfaces are embedded punctured spheres or tori, explicit bounds are known on the number of boundary slopes. These bounds are based on highly developed combinatorial methods in knot theory and the theory of representations of knot groups. See the survey papers [Go], [Lu] and [Sh].

(3) When ∂M is a torus and the surfaces are immersed, a positive answer to Question (1) has been obtained recently in [HRW]. When M is hyperbolic, minimal surface theory is used to derive these bounds. For fixed genus g, these turn out to be quadratic functions of g, independent of M. See also recent work of Agol [Agol].

(4) If M is an irreducible, ∂-irreducible, acylindrical, atoroidal 3-manifold and the surfaces are embedded, Scharlemann and Wu [SW] gave a positive answer to Question 1 using combinatorial arguments.

(5) Suppose ∂M is a torus and the surfaces are immersed. Baker has given examples to show that the bounded genus assumption cannot be dropped. Oertel, using branched surface theory, has found manifolds in which every slope is realized by the boundary of an immersed essential surface [Oe].

In this note we give a positive answer to Question (1), which extends (3) to the case where ∂M can contain high genus components and generalizes (4) from embedded to immersed surfaces.

Theorem 1. *Suppose M is ∂-irreducible, acylindrical, atoroidal 3-manifold. Then for any g, there are only finitely many ∂-slopes for essential surfaces of genus g.*

Next we consider the question of obtaining bounds for the number of possible slopes which are independent of the particular manifold we are studying. It turns out that only the genus of the boundary of M is relevant.

We define the genus of ∂M be the sum of the genus of the components of ∂M.

Theorem 2. *There is a function $n(g, g_\partial)$ such that there are at most $n(g, g_\partial)$ ∂-slopes for essential surfaces of genus g in a ∂-irreducible, acylindrical, atoroidal 3-manifold whose boundary has genus equal to g_∂.*

We can also obtain bounds on the number of boundary slopes in terms of hyperbolic geometry.

Definition. Let $\mathcal{M}(V)$ be the set of all hyperbolic 3-manifolds of totally geodesic boundary and with volume bounded above by $V > 0$.

Theorem 3. *There is a function $n_1(g, V)$ such that there are at most $n_1(g, V)$ ∂-slopes for essential surfaces of genus g in a 3-manifold $M \in \mathcal{M}(V)$.*

Theorem 3 follows from either Theorem 1 or Theorem 2, the fact that all maximum torus cusps have volumes $> C > 0$ [Ad], and the following

Theorem 4. *There is an integer $g^* > 0$ and a number $L > 0$ such that if $M \in \mathcal{M}(V)$, then*
(1) the genus of ∂M is at most g^*.
(2) the length of any closed geodesic on ∂M is at least L.

Remark on Theorem 4. Theorem 4 can be restated as follows: For hyperbolic 3-manifolds with totally geodesic boundary and bounded volume, the areas of their boundaries have an upper bound, and the lengths of simple closed geodesics on their boundary have a lower bound. Neither of those two assertions is true in dimension 2. Surfaces of given area can have geodesic boundaries of any length.

Proof of Theorem 1. If M has any 2-sphere boundary components, we can fill them in with balls without changing the number of boundary slopes. Since any essential surface can be homotoped off of a splitting 2-sphere, we can without loss of generality assume that M is irreducible. The number of boundary slopes of essential surfaces lying on a torus boundary component of M is finite by [HRW], so we restrict attention to surfaces with boundary on a higher genus component of ∂M. By Thurston's Geometrization Theorem for Haken manifolds, M admits a complete hyperbolic structure of finite volume with totally geodesic boundary [T]. We assume that M is equipped with such a hyperbolic structure. The totally geodesic boundary components consist of the non-torus boundary components of M. Since ∂M have only finitely many components, to prove Theorem 1, we need only to show that for each component of ∂M there are finitely many boundary slopes of proper essential surfaces of genus at most g.

Suppose F is an incompressible, boundary incompressible proper immersion with ∂F consisting of n copies of a slope l. Let DM be the double of M along its totally geodesic boundary components. DM is Haken and atoroidal, and admits a hyperbolic structure obtained by doubling that of M. The double DF of F is incompressible, and therefore a theorem of Schoen-Yau and Sacks-Uhlenbeck shows that there is a least area representative of its homotopy class, denoted by DF^* [SY]. The intersection of DF^* with the incompressible least area (in fact totally geodesic) surface ∂M consists of curves essential on both DF^* and ∂M. Since F is boundary incompressible in M, the intersection F^* of DF^* with M is a surface homotopic (rel boundary) in M to F. Since DM admits an isometry which is a reflection about ∂M, DF^* is perpendicular to ∂M. If not, we could reflect $DF^* \cap M$ and get a homotopic surface with lower area. So F^* is properly homotopic to F, F^* is perpendicular to ∂M and ∂F^* is a (possibly multiply covered) geodesic.

Choosing geodesic orthogonal coordinates near the geodesic boundary of the surface F^*, we have (line 7 of p.374, [BM])

$$(1) \qquad ds^2 = du^2 + J^2(u,v)dv^2, \qquad (J(u,v) > 0 \text{ and } J(0,v) = 1).$$

where the u-curves (those where $v =$ constant) are geodesics perpendicular to the boundary and the v-curves lie on the boundary when $u = 0$.

The geodesic curvature in F^* of a curve $t \mapsto (u(t), v(t))$ is given by Formula 10.4.7.1 of [BM],

$$(2) \qquad \frac{1}{\sqrt{Eu'^2 + Gv'^2}}\left(\frac{d\phi}{dt} + \frac{1}{2\sqrt{EG}}\left(\frac{\partial G}{\partial u}v' - \frac{\partial E}{\partial v}u'\right)\right),$$

where ϕ is the angle between the curve and the u-curves and the metric on F^* is given by

$$Edu^2 + Gdv^2.$$

When we consider the v-curves, we have $u' = 0$, $v' = 1$, $\phi = \pi/2$, $E = 1$ and $G = J^2$. Substituting into (2), the geodesic curvature for a v-curve $\{u = c\}$ oriented as the boundary of $\{0 \le u \le c\}$ is given by:

$$(3) \qquad k_g = \frac{1}{J}\frac{\partial J}{\partial u}.$$

Orienting the curve as the boundary of $\{u \ge c\}$ changes the sign and gives

$$(3') \qquad k_g = -\frac{1}{J}\frac{\partial J}{\partial u}.$$

The Gaussian curvature of the surface is Formula 10.5.3.3 of [BM]

$$(4) \qquad K = -\frac{1}{J}\frac{\partial^2 J}{\partial u^2}.$$

A direct computation shows that k_g satisfies the following equation

$$\frac{\partial k_g}{\partial u} = K + k_g^2. \tag{5}$$

Since M is of constant curvature -1, we have

$$K = k_1 k_2 - 1$$

by Gauss's Formula (p.179 [Sp]), where k_1 and k_2 are the principle curvatures. Since F is a minimal surface, we have $k_1 k_2 \le 0$, and hence $K \le -1$. Then by (4) it follows that

$$\frac{\partial^2 J}{\partial u^2} \ge J. \tag{6}$$

Fixing $v = v_0$, by (6) we have

$$\frac{\partial J}{\partial u}(u, v_0) = \frac{\partial J}{\partial u}(u, v_0) - \frac{\partial J}{\partial u}(0, v_0)$$

$$= \int_0^u \frac{\partial^2 J}{\partial u^2}(s, v_0) ds \ge \int_0^u J(s, v_0) ds \ge 0. \tag{7}$$

(1), (3') and (7) imply that $k_g < 0$, if $u > 0$.

Now consider the function

$$h(u) = \frac{e^{-u} - e^u}{e^{-u} + e^u}. \tag{8}$$

which is the solution to the differential equation

$$\frac{dh}{du} = -1 + h^2 \tag{9}$$

with the initial condition $h(0) = 0$. Note that $h(u) < 0$ when $u > 0$ and that the function $k_g - h$ satisfies the differential inequality

$$\frac{d(k_g - h)}{du} = K + k_g^2 + 1 - h^2$$

$$\leq k_g^2 - h^2 = (k_g - h)(k_g + h) \tag{10}$$

by (5) and (9). We want to show that $k_g - h \leq 0$.

Suppose on the contrary that on an interval $[0, U]$, $k_g - h$ is somewhere positive. Pick a $u_0 \in [0, U]$, such that $k_g - h$ takes its positive maximum at u_0. We know $u_0 \neq 0$ since ∂F^* is a geodesic and $k_g = h = 0$ at 0. Then

$$\frac{d(k_g - h)}{du}$$

is zero if $u_0 \in (0, u)$, and is ≥ 0 if $u_0 = U$. Hence

$$\frac{d(k_g - h)}{du} \geq 0$$

at u_0. Since both k_g and h are negative at u_0, we have

$$(k_g - h)(k_g + h) < 0.$$

This contradicts (10), and so $k_g \leq h$.

For $t > 0$, let $N_t(\partial M)$ be the subset of M with distance $\leq t$ from the boundary. There is a $b > 0$ such that when $t < b$ then $N_t(\partial M)$ is a collar of ∂M.

Choose $U < b$ in the above and let $N_U(\partial F^*)$ be the neighborhood of ∂F^* with u coordinates at most U. Clearly $N_U(\partial F^*) \subset N_b(\partial M)$. Since $N_b(\partial M)$ is a collar of ∂M and the surface F^* is ∂-incompressible, it follows that $N_U(\partial F^*)$ is a collar of ∂F^*. Letting

$$F_U = \overline{F^* - N_U(\partial F^*)},$$

each component of ∂F_U is in the same homotopy class in M as the slope l and

$$\#\partial F_U = \#\partial F^* = \#\partial F = n.$$

By Gauss-Bonnet, we have that

$$\int_{F_U} K dA + \int_{\partial F_U} k_g ds = 2\pi(\chi(F)) = 2\pi(2-2g-n).$$

Let d be the length of the geodesic in the homotopy class of the slope l. Then the length of each component of ∂F_U is larger than d. Since $K \le -1$ and $k_g \le h < 0$ at U, we have

$$nhd \ge 2\pi(2-2g-n).$$

Then we have

$$d \le \frac{2\pi(2g+n-2)}{-hn} \le \frac{2\pi(2g+1)}{-h}. \tag{11}$$

Since g is given and $h = h(U) < 0$, d is bounded above. There are only finitely many homotopy class of essential closed curves in ∂M containing elements of length less than a given constant. Therefore for any fixed g there are only finitely many ∂-slopes for immersed incompressible, boundary incompressible surfaces of genus g. $\square$

Proof of Theorem 2. In the proof of Theorem 1, if we only consider 3-manifolds whose totally geodesic boundary has a collar of width bounded below by U_*, so that $U > U_*$, then $h = h(U) \le h(U_*) = -\tanh U_* \le 0$. Moreover if we consider only boundary slopes of length at least $L > 0$, then by (11) we have

$$L \le d \le \frac{2\pi(2g+1)}{\tanh U_*}. \tag{12}$$

Let $A(R)$ be the area of $D(R)$, the hyperbolic disc with radius R. Let Γ_L be any lattice on the hyperbolic plane such that the distance of any two vertices has distance at least L. Then the number of vertices of Γ_L in $D(\frac{2\pi(2g+1)}{\tan U_*})$ is at most

$$n(g, U_*, L) = \frac{A(\frac{2\pi(2g+1)}{\tan U_*} + L)}{A(L)}. \tag{13}$$

It follows that the number of boundary slopes for proper essential surfaces of genus at most g is bounded by $n(g, U_*, L)$.

To show the existence of the function $n(g, g_\partial)$ we need to establish in the proof of the previous theorem:

1. A lower bound U_* to the width of a collar around ∂M for any hyperbolic metric on a manifold M in which ∂M is totally geodesic of genus $\leq g_\partial$.

2. Given $L > 0$, an upper bound on the number of curves of length $\leq L$ lieing in a collar of ∂M. This bound should depend only on the genus of ∂M, and not on its geometry.

The existence of the first type of bound was established by Kojima and Miyamoto [KM], and by Basmajian [Ba]. On the boundary of M, the second type of bound is a consequence of the Margulis Lemma, or of its two dimensional version known as the "collar lemma" ([Bu] and also Theorem 2.18 of [Mu]). We actually use a bound that holds in a collar neighborhood of ∂M in Theorem 1. However the projection from a collar of the boundary of a hyperbolic manifold with totally geodesic boundary to the boundary is length decreasing, so it suffices to consider curves lieing on the boundary.

More precisely, let $S(x) = \sinh^{-1}(1/\sinh(x/2))$. For a given simple closed geodesic c with length d_c on a hyperbolic surface, let $N(c) = \{x : d(x, c) \leq S(d_c)\}$. Then the collar lemma states that $N(c)$ is a collar. Moreover if c_1 and c_2 are disjoint simple closed geodesics, then $N(c_1)$ and $N(c_2)$ are disjoint. There is an L such that if $d \leq L$, then $S(d) > d/2$ and $d > \sinh(d/2)$; for example, we can choose $L = 1.75$; then $S(d) > S(L) > 0.887 > 0.85 = L/2 \geq d/2$ and $1.76 \leq L/\sinh(L/2) \leq d/\sinh(d/2)$. Then any two simple closed geodesics of length $\leq L$ are disjoint. Moreover the area of $N(c)$ is

$$2d_c \sinh(S(d_c)) = 2d_c/\sinh(d_c/2) \geq 2d_c/d_c = 2.$$

Hence the number of simple closed geodesics of length at most L is bounded above by

$$(14) \qquad 2\pi(2g(F)-2)/2 = 2\pi(g(F)-1).$$

where $g(F)$ is the genus of F.

For simplicity, we first assume that ∂M is connected. By (13) and (14) we have

$$(15) \qquad n(g, g_\partial) = \frac{A(\frac{2\pi(2g+1)}{\tanh U^*} + L)}{A(L)} + 2\pi(g_\partial - 1).$$

By Lemma 3.1 of [Ba], we have the lower bound

$$(16) \qquad U^* = \frac{1}{4} log \frac{g_\partial + 1}{g_\partial - 1}.$$

Moreover $A(R) = \dfrac{4\pi}{1 - \tanh^2 R/2}$. We can get an explicit value for $n(g, g_\partial)$ by plugging in these functions, though this does not appear to give sharp values.

In general suppose ∂M consists of k torus components and l components of genus $g_i > 1$, $i = 1, ..., l$. Then $g_\partial = \sum_{i=1}^{l} g_i + k$ and there are at most $\sum_{i=1}^{l} n(g, g_i) + kN(g)$ boundary slopes for proper essential surfaces of genus g, where $N(g)$ is the uniform bound for the number of boundary slopes of proper essential surface of genus g on a torus boundary component given in [HRW], and $n(g, g_i)$ is given by (15). One can verify that $\sum_{i=1}^{l} n(g, g_i) + N(g) \leq n(g, g_\partial)$. $\square$

Proof of Theorem 4. Pick any infinite sequence of totally geodesic hyperbolic 3-manifolds $\{M_n\}$ in $\mathcal{M}(V)$. Consider the sequence $\{D(M_n)\}$, where $D(M_n)$ is the double of each M_n. Then the volume of the closed hyperbolic 3-manifold $D(M_n)$ is bounded by $2V$. By passing to a subsequence, we can assume that $D(M_n)$ has a Gromov limit M^*. It is known that

(a) M^* is a complete hyperbolic 3-manifold of finite volume, which can be viewed as the complement of a hyperbolic link L in a closed 3-manifold, and each $D(M_n)$ is obtained by a Dehn surgery on M^*.

(b) Since each $D(M_n)$ admits a reflection r_n (isometry) about its geodesic boundary, so does M^*. Hence $M^* = D(M_\infty)$, where M_∞ is a hyperbolic 3-manifold with totally geodesic boundary. Let r_∞ be the reflection of $D(M_\infty)$ about ∂M_∞. We have not claimed as yet that there is no cusp at ∂M_∞.

(c) Let $TH_\epsilon(P)$ be the ϵ thick part of P for any hyperbolic 3-manifold P. Then for any $\epsilon > 0$ and $1 - \epsilon \leq k \leq 1$, there is an integer N such that for $n > N$ there is a homeomorphism $h_n : TH_\epsilon(D(M)) \to TH_\epsilon(D(M_n))$ which is a k-quasi-isometry. Moreover h_n can be chosen to commute with the reflections.

For the result on the Gromov limit of closed hyperbolic 3-manifolds of bounded volume, see Chapter 6 of [T1], or Chapter E of [BP]. For the fact about reflections, one can argue as follow: As in the case of closed hyperbolic 3-manifolds, any sequence of hyperbolic 3-manifolds with totally geodesic boundary and bounded volume V has a subsequence with Gromov limit M_∞, which is a complete hyperbolic 3-manifold with totally geodesic boundary. Then the double $D(M_\infty)$ will be the limit of the doubles.

Suppose there is a torus in ∂M, which must result in cusps in ∂M_∞. Let the torus T be the boundary component of $TH_\epsilon(M_\infty)$ corresponding to the cusp C, and let c be a component of $T \cap \partial M$. Then $h_n(T) \subset D(M_n)$ is invariant under the reflection r_n about ∂M_n, and it follows that $h_n(c)$ is a meridian of the Dehn filling solid torus on $h_n(T)$, and therefore $h_n(c) \subset \partial M_n$ is a trivial loop. However each cusp in ∂M_∞ can only be a limit of essential loops, and this is a contradiction. Hence ∂M_∞ contains no cusps.

Since ∂M_∞ contains no cusps, for small ϵ, ∂M_∞ is contained in the interior of the compact manifold $TH_\epsilon D(M_\infty)$. Moreover as the fixed point set of the reflection $r_\infty | TH_\epsilon(D(M_\infty))$, ∂M_∞ is compact, therefore it is closed. Since ∂M_n

converges to ∂M_∞ in the limit, it follows that

(1') the genus of ∂M_n is stable when n is large enough,

(2') the length of the shortest simple closed geodesic on ∂M_n cannot converge to zero (otherwise there will be a cusp in M_∞).

Now suppose (1) of Theorem 4 is not true. Then we can find a sequence $\{M_n\}$ in $\mathcal{M}(V)$ such that the genus of ∂M_n is $> n$. The genus of any subsequence must also tend to infinity, which contradicts (1'); hence (1) of Theorem 4 is true. Similarly, if (2) of Theorem 4 is not true, then we can find a sequence $\{M_n\}$ in $\mathcal{M}(V)$ such that the length of the shortest geodesic of ∂M_n is $< 1/n$, which contradicts (2'). This finishes the proof of Theorem 4. $\square$

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