

STABILIZATION OF HEEGAARD SPLITTINGS

JOEL HASS, ABIGAIL THOMPSON, AND WILLIAM THURSTON

ABSTRACT. For each $g \geq 2$ there is a 3-manifold with two genus g Heegaard splittings that require g stabilizations to become equivalent. Previously known examples required at most one stabilization. Control of families of Heegaard surfaces is obtained through a deformation to harmonic maps.

1. INTRODUCTION

While area minimizing surfaces have proved to be a powerful tool in the study of 3-dimensional manifolds, harmonic maps of surfaces to 3-manifolds have not been as widely applied, due to several limitations. A homotopy class of surfaces generally gives rise to a large space of harmonic maps. A harmonic map of a surface need not minimize self-intersections and may fail to be immersed. In negatively-curved 3-manifolds there is a unique harmonic map in each conformal class of metrics on the domain, and smooth families of surfaces give rise to smooth families of harmonic maps [5],[7]. In this paper we study Heegaard splittings of 3-manifolds using families of harmonic surfaces.

A *genus g Heegaard splitting* of a 3-manifold M is a decomposition of M into two genus g handlebodies with a common boundary. It is described by an ordered triple (H_1, H_2, S) where each of H_1, H_2 is a handlebody and the two handlebodies intersect along their common boundary S , called a *Heegaard surface*. Two Heegaard splittings (H_1, H_2, S) and (H'_1, H'_2, S') of M are equivalent if an ambient isotopy of M carries (H_1, H_2, S) to (H'_1, H'_2, S') . Every 3-manifold has a Heegaard splitting [14], and Heegaard splittings form one of the basic structures used to analyze and understand 3-manifolds.

Corresponding to the Heegaard splitting is a family of surfaces that sweep out the manifold, starting with a core of one handlebody and ending at a core of the second. This family is geometric controlled by deforming it to a family of harmonic maps. When the manifold is negatively curved, harmonic maps of genus g surfaces have uniformly bounded area. In the manifolds we consider, the geometry forces small area surfaces to line up with small

Date: March 7, 2008.

1991 Mathematics Subject Classification. Primary 57M25.

Key words and phrases. Heegaard splitting, stabilization, harmonic map, isoperimetric inequality.

All three authors were partially supported by NSF grants.

area cross sections of the manifold. As a result we obtain obstructions to the equivalence of distinct Heegaard splittings.

A *stabilization* of a genus g Heegaard surface is a surface of genus $g + 1$ obtained by adding a 1-handle whose core is parallel to the surface. Such a surface splits the manifold into two genus $g + 1$ handlebodies, and thus gives a new Heegaard splitting. Two genus g Heegaard splittings are *k-stably equivalent* if they become equivalent after k stabilizations. Any two Heegaard splittings become equivalent after a sequence of stabilizations [23]. An upper bound on the number of stabilizations needed to make two splittings equivalent is known in some cases. If G_p and G_q are splittings of genus p and q with $p \leq q$, and M is non-Haken, then Rubinstein and Scharlemann obtained an upper bound of $5p + 8q - 9$ for the genus of a common stabilization [18]. For all previously known examples of manifolds with distinct splittings, the splittings become equivalent after a single stabilization of the larger genus Heegaard surface. The question of whether a single stabilization always suffices is sometimes called the *stabilization conjecture* [9] (Problem 3.89), [20], [22], [21]. In Section 7 we show that this conjecture does not hold. There are pairs of genus g splittings of a 3-manifold that require g stabilizations to become equivalent.

Theorem 1.1. *For each $g > 1$ there is a 3-manifold M_g with two genus g Heegaard splittings that require g stabilizations to become equivalent.*

We outline the idea in Section 2. In Section 3 we describe the construction of the 3-manifolds M_g . We derive isoperimetric inequalities used in the proof in Section 4 and discuss deformations of surfaces to harmonic maps in Section 5. In Section 6 we show how a Heegaard splitting gives rise to a family of Heegaard surfaces that sweep out the 3-manifold. The proof of Theorem 1.1 is given in Section 7. Finally in Section 8 we show how a somewhat weaker result can be obtained for an easily constructed class of hyperbolic manifolds.

This result was presented at the American Institute of Mathematics Conference on *Triangulations, Heegaard Splittings, and Hyperbolic Geometry* held in December 2007. At this conference D. Bachman announced, using different methods, examples giving a lower bound of $g - 4$ for the number of required stabilizations.

2. OUTLINE OF THE ARGUMENT

Let M_ϕ be a hyperbolic 3-manifold that fibers over S^1 with monodromy ϕ and let $\tilde{M}_\phi$ denote its infinite cyclic cover. A pictorial representation of M_ϕ is given in Figure 1. Cutting open M_ϕ along a fiber gives a fundamental domain B of the $\mathbb{Z}$ -action on the infinite cyclic cover, which we call a *block*. Blocks are homeomorphic, but not isometric, to the product of a surface and an interval. They are foliated by fibers of M_ϕ , which in B we call *slices*. By cutting open a cyclic cover of M_ϕ we obtain a hyperbolic 3-manifold with as many adjacent blocks as we wish.

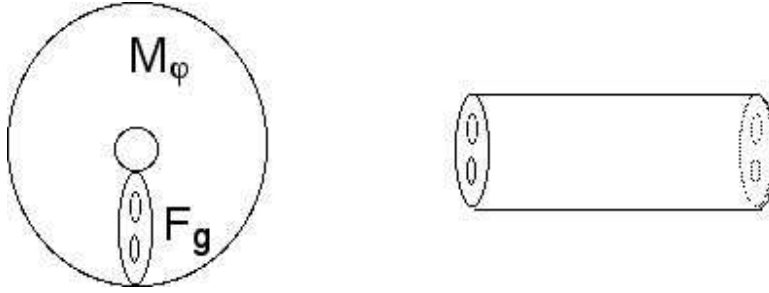


FIGURE 1. M_ϕ and one block B , a fundamental domain of the $\mathbb{Z}$ -action on its infinite cyclic cover. B is homeomorphic to $F_g \times I$.

The manifold M_g used in our main result has pinched negative curvature. It contains two handlebodies H_L and H_R with fixed Riemannian metrics, separated by a region homeomorphic to the product of a surface of genus g with an interval. This intermediate piece is hyperbolic and isometric to $2n$ adjacent blocks. The first n blocks form a submanifold called L and the next n form a submanifold called R , as in Figure 2. The value of n can be chosen as large as desired without changing the geometry of H_L and H_R . Details are in Section 3.

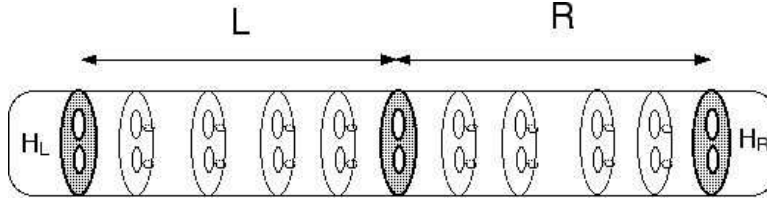


FIGURE 2. M_2 , constructed with 10 blocks, 5 in each of L and R .

M_g has two obvious Heegaard splittings $E_0 = (H_L \cup L, H_R \cup R, S)$ and $E_1 = (H_R \cup R, H_L \cup L, -S)$, where S is a surface of genus g separating $H_L \cup L$ and $H_R \cup R$ and $-S$ indicates S with reversed orientation. We will show that these splittings are not k -stably equivalent for $k < g$.

Let G_0 be the Heegaard splitting obtained by stabilizing E_0 $(g-1)$ times and G_1 be the Heegaard splitting obtained by stabilizing E_1 $(g-1)$ times. If G_0 and G_1 are equivalent, then there is an isotopy $\{I_s, 0 \leq s \leq 1\}$ of M_g that induces a diffeomorphism of M_g carrying G_0 to G_1 . A family of Heegaard splittings $\{G_s = I_s(G_0), 0 \leq s \leq 1\}$ interpolates between G_0 and G_1 .

Associated to each Heegaard splitting G_s is a family of surfaces $F_{s,t}$ sweeping out M_g from one core to the other. In Section 5 we show that such a family of surfaces can be deformed to a family of harmonic, or energy minimizing, maps. These harmonic surfaces have area uniformly bounded by

a constant A_0 . This area bound restricts the way that surfaces can divide up the volume of M_g . The surface cannot simultaneously split in half the volumes of L and $L \cup R$. In Section 7 we use this to show that surfaces in such bounded area families cannot interpolate between G_0 and G_1 . We conclude that E_0 and E_1 are not k -stably equivalent for $k < g$.

Remark. The two Heegaard splittings E_0 and E_1 of M_g do become equivalent after g stabilizations. After adding g trivial handles, a Heegaard surface can be isotoped to a surface formed by taking two genus g slices connected by a thin tube. This can be isotoped so that one slice splits the volume of L in half while the second does the same to R , and together they bisect the volume of $L \cup R$. This type of surface arises in a family of bounded area surfaces interpolating between genus $2g$ stabilizations of E_0 and E_1 .

3. CONSTRUCTION OF M_g

In this section we describe a certain Riemannian 3-manifold M_g of Heegaard genus g , based on the work of Namazi and Souto [12]. In Section 8 we give a simpler construction of a hyperbolic manifold that gives slightly weaker lower bounds on the number of stabilizations needed to make two Heegaard splittings equivalent.

We begin with a hyperbolic manifold M_ϕ that fibers over the circle with fiber a genus g surface. Fix a fibration of M_ϕ with fibers $\{S_t, 0 \leq t \leq 1\}$ satisfying $\phi(S_0) = S_1$. Define a *block* B to be the manifold obtained by cutting open M_ϕ along S_0 and a *block manifold* B_n to be a union of n blocks, formed by placing end to end n copies of B . The manifold B_n is topologically, though not geometrically, the product of a surface of genus g and an interval. Its geometry can be obtained by cutting open the n -fold cover of M_ϕ along a lift of S_0 . We call the fibers of B_n *slices* and label them by S_t , $0 \leq t \leq n$. We label the union of all fibers between S_{t_1} and S_{t_2} by $B[t_1, t_2]$.

The properties we desire for the manifold M_g are that it is a union of four pieces as in Figure 2. Two pieces are genus g handlebodies, H_L and H_R , and the other two L and R are each homeomorphic, though not isometric, to a product $F_g \times [-1, 1]$, where F_g is a surface of genus g . L and R are each hyperbolic, and isometric to a block manifold B_n . The sectional curvatures of M_g are pinched between $-1 - \epsilon_0$ and $-1 + \epsilon_0$, where $\epsilon_0 > 0$ can be chosen arbitrarily small. For our purpose we take $\epsilon_0 = 1/2$.

Namazi-Souto produced manifolds very similar to M_g [12]. While the manifolds they construct are suitable for the constructions we give, the argument is simpler if we modify the metric slightly so that the middle part of M_g is precisely, rather than approximately hyperbolic. We require the metric on M_g to be isometric to a union of blocks on $L \cup R$. This can be arranged by perturbing the manifold M in Theorem 5.1 of [12], which has sectional curvatures pinched between $-1 - \epsilon$ and $-1 + \epsilon$ where $\epsilon > 0$ can be chosen arbitrarily small. This manifold has a product region separating

two handlebodies with geometry ϵ close (in the C^2 -metric) to the geometry of a block. A small C^2 -perturbation gives a new metric on M with sectional curvatures between $-3/2$ and $-1/2$ everywhere and exactly isometric to a block on an interval separating two handlebodies. Now cut open the manifold along a slice in this hyperbolic block and insert a copy of B_n to obtain M_g , with n as large as desired. However large the choice of n , the geometry of M_g is unchanged on the two complementary handlebodies of B_n , which we call H_L and H_R .

4. ISOPERIMETRIC INEQUALITIES

In this section we develop some isoperimetric inequalities for curves in surfaces and for surfaces in 3-manifolds. The following isoperimetric inequality holds for a curve in hyperbolic space H^2 [2]. A proof can be obtained by using symmetry to show that a round circle bounds at least as much area as any other curve, using no more length. Explicit formulas for length and area in hyperbolic space then imply the inequality. The curve does not need to be embedded or connected.

Lemma 4.1. *Let c be a closed curve in H^2 with length L . Then c is the boundary of a disk $f : D^2 \rightarrow H^2$ of area A , with $A < L$.*

We now consider more general isoperimetric inequalities for curves in surfaces and surfaces in 3-manifolds. In these settings we consider the areas and volumes of 2-chains and 3-chains with boundary. We first give an isoperimetric inequality for curves in a compact family of Riemannian metrics on a surface.

Lemma 4.2. *Let (S, g_r) be a closed 2-dimensional surface with Riemannian metric g_r , $r \in U$, where g_r is a family of metrics smoothly parameterized by a compact set $U \subset \mathbb{R}^n$. There is a constant K such that for any surface S in the family (S, g_r) ,*

- (1) *A null-homotopic curve c in S is the boundary of a disk $f : D \rightarrow S$ with*

$$\text{Area}(f(D)) \leq K \cdot \text{Length}(c).$$

- (2) *A collection of curves $\{c = \cup c_i\}$, with each c_i null-homotopic in S , is the boundary of a collection of disks $\{f_i : D_i \rightarrow S\}$ with*

$$\sum \text{Area}(f_i(D_i)) \leq K \cdot \text{Length}(c).$$

- (3) *A null-homologous curve c on S bounds a 2-chain X_2 in S with*

$$\text{Area}(X_2) \leq K \cdot \text{Length}(c).$$

Proof. First consider the case where c is a null-homotopic curve. For a hyperbolic surface S , c lifts to H^2 where it bounds a mapped in disk of area less than $\text{length}(c)$ by Lemma 4.1. The surface (S, g_r) is conformally equivalent to a hyperbolic surface, so there is a hyperbolic metric (S, g) and a diffeomorphism from (S, g) to (S, g_r) that stretches or compresses any

vector in the tangent space of (S, g) by a factor of at most λ . A single choice of λ can be made for all the surfaces in the compact family (S, g_r) . Lengths of curves measured in (S, g) differ from lengths in (S, g_r) by a factor of at most λ and the area of a region in (S, g) differs from the area in (S, h_r) by a factor of at most λ^2 . By comparing the area of a spanning disk and $\text{Length}(c)$ in (S, g) , we see that in (S, g_r)

$$\text{Area}(f(D)) \leq \lambda^3 \cdot \text{Length}(c).$$

The same argument applies for a collection of null-homotopic components,

There is a constant ϵ that gives a lower bound for the injectivity radius for the family of metrics h_r . If the length of c is less than ϵ than each of the components of c is null-homotopic and the above bound holds for the area of a collection of spanning disks.

Now consider a general null-homologous curve c of length greater than ϵ . A small perturbation and orientation preserving cut and paste transforms c into a union of disjoint embedded components that is homologous to c . A homology between this embedded curve and c has arbitrarily small area, so we can assume that c is a collection of embedded disjoint curves. Each side of c gives a 2-chain with boundary c , and so $\text{Area}(S)$ gives an upper bound to the area of a 2-chain X_2 spanning c . Therefore

$$\text{Area}(X_2) \leq \text{Area}(S) \leq (\text{Area}(S)/\epsilon) \cdot \text{Length}(c).$$

The Lemma follows with $K = \lambda^3 + \text{Area}(S)/\epsilon$. $\square$

To simplify the next calculations, it is convenient to work with chains having $\mathbb{Z}_2$ -coefficients. With this choice we do not need to pay attention to orientations, signs, or multiplicity other than even or odd. A 2-chain spanning a general position curve in a surface is obtained by 2-coloring the complement of the curve, dividing the surface into black and white regions separated by the curve. The area of a spanning 2-chain is the area of either the white or the black subsurface, and has a value between zero and the area of the surface. The two choices of spanning 2-chain have area that sums to the area of the entire surface. A similar statement applies for 3-chains spanning a surface in a 3-manifold.

We now analyze the geometry of surfaces in block manifolds. The metrics on the slices S_t of a block manifold, up to isometry, are parameterized by a closed interval, so Lemma 4.2 applies. Therefore there is a constant K_1 independent of n , such that a null-homologous curve c on a slice S_t bounds a 2-chain X_2 in that slice with

$$\text{Area}(X_2) \leq K_1 \cdot \text{Length}(c).$$

Each slice is an incompressible surface in B_n and therefore with its induced metric has injectivity radius greater or equal to that of M_ϕ . Set ϵ to be the constant

$$\epsilon = \min_{t \in [0,1]} \{\text{injectivity radius}(M_\phi), \text{Area}(S_t)/10K_1\}$$

Now consider a proper separating surface $(F, \partial F)$ in $(B_n, \partial B_n)$. For each t , the complement of $F \cap S_t$ in the slice S_t can be 2-colored. In slices S_t where $\text{Length}(F \cap S_t) \leq \epsilon$, each curve of $F \cap S_t$ is null-homotopic. Lemma 4.2 implies that there is a collection of disks spanning the components of $F \cap S_t$ whose total area is less than $\text{Area}(S_t)/10$. Call the subsurface of S_t missing these disks the *large side* of $F \cap S_t$. Alternately, this subsurface is the largest of the connected complementary components of $F \cap S_t$ in S_t .

Lemma 4.3. *Suppose $(F, \partial F)$ is a proper separating surface in $(B_n, \partial B_n)$ and intersects slices S_{t_1}, S_{t_2} transversely with $\text{Length}(F \cap S_{t_1}) \leq \epsilon$ and $\text{Length}(F \cap S_{t_2}) \leq \epsilon$. If the genus of F is less than g then an arc α from the large side of $F \cap S_{t_1}$ to the large side of $F \cap S_{t_2}$ intersects F algebraically zero times.*

Proof. Cap off the curves in $F \cap S_{t_1}, F \cap S_{t_2}$, each of which is shorter than the injectivity radius, by adding disks missing the large side of each surface. These disks do not intersect the arc α . The resulting closed surface in the region between S_{t_1} and S_{t_2} , has genus less than g . Such a surface is null-homologous in the region between these two slices, which is homeomorphic to the product of a genus g surface and an interval, and therefore cannot algebraically separate them. $\square$

The next lemma gives a 3-dimensional isoperimetric inequality for surfaces in block manifolds. Note that Lemma 4.4 is false if the genus $k \geq g$. A slice can of genus g can split the volume of B_n in half.

Lemma 4.4. *Let $(F, \partial F) \subset (B_n, \partial B_n)$ a null-homologous proper surface in a block manifold, having genus less than g and area less than some constant A_0 . Then there is a constant B_0 , independent of n , such that F bounds a 3-chain X in B_n with $\text{Volume}(X) < B_0$. For n sufficiently large,*

$$\text{Volume}(X) < (.1) \text{Volume}(B_n).$$

Proof. Since F is null-homologous, we can 2-color its complement. This gives two 3-chains with boundary F , and we take X to be the one of smaller volume. The coarea formula [6] implies that the volume of X is comparable to the integral of its cross-sectional areas. Namely there is a constant C_1 such that

$$(1) \quad \text{Volume}(X)/C_1 < \int_0^n \text{Area}(X \cap S_t) dt < C_1 \cdot \text{Volume}(X).$$

The coarea formula also gives an inequality between the area of F in B_n and the lengths of its intersections with slices. There is a constant C_2 such that

$$(2) \quad \int_0^n \text{Length}(F \cap S_t) dt < C_2 \cdot \text{Area}(F).$$

The constants C_1 and C_2 are determined by the geometry of a single block B , and do not depend on n .

We estimate the volume of X by integrating the area of its cross sections. To do so, we need to consistently specify for each slice S_t which of the two complementary subsurfaces of $F \cap S_t$ lies in X . If $\text{Length}(F \cap S_t) < \epsilon$ then we take the side of F containing the complementary subsurface of smaller area in S_t . This subsurface has area less than $K_1 \epsilon$ by Lemma 4.2 and misses the large side of $F \cap S_t$. Each such subsurface lies on the same side of F (mod 2) by Lemma 4.3. For slices in which $\text{Length}(F \cap S_t) \geq \epsilon$ the construction given in Lemma 4.2 gives a bound $K_1 \cdot \text{Length}(F \cap S_t)$ for the area of both complementary sides in S_t of $F \cap S_t$, and in particular for the side lying in X . By Equation (1), X has volume bounded above by

$$\begin{aligned} \text{Volume}(X) &< C_1 \int_0^n \text{Area}(X \cap S_t) dt < C_1 K_1 \int_0^n \text{Length}(F \cap S_t) dt \\ &< C_1 K_1 C_2 A_0 = B_0. \end{aligned}$$

This bound is independent of n , so for n sufficiently large it is less than $\text{Volume}(B_n)/10$. $\square$

Finally we give a 3-dimensional isoperimetric inequality that implies small area surfaces in a Riemannian manifold bound regions of small volume. The argument follows the line of an argument of Meeks [13]. The result is valid for both integer and $\mathbb{Z}_2$ coefficients, though we need only the latter.

Lemma 4.5. *Let M be a compact, connected Riemannian 3-manifold and F a proper surface in M , not necessarily embedded or connected. Given $\epsilon_0 > 0$ there is a $\delta_0 > 0$ such that if $\text{Area}(F) < \delta_0$ then F bounds a 3-chain X in M with $\text{Volume}(X) < \epsilon_0$.*

Proof. By cut and paste we can reduce to the case where F is embedded, though not necessarily connected. The result is true for a surface in the unit ball in $\mathbb{R}^3$, where we have the inequality

$$V \leq \frac{(A)^{3/2}}{6\sqrt{\pi}}.$$

for the area of a surface A and the volume V of the region it bounds.

For a fixed Riemannian metric on a ball, there is a bound λ to the maximum stretch or compression of the length of a vector. This implies a bound on the volume increase of a region of λ^3 and a bound on the change in the area of a surface of λ^2 . It follows that

$$(3) \quad V \leq C \cdot A^{3/2}$$

for some constant C depending only on the metric.

Now take a cell decomposition of M with a single 3-cell R . By the coarea formula and the area bound for F , for sufficiently small δ_0 the 2-complex forming the boundary of this 3-cell ∂R can be perturbed so that the length of its intersection with F is arbitrarily short. The isoperimetric inequality for the boundary 2-sphere of R allows us to surger F along short intersection curves with the boundary, while keeping its total area less than a constant δ_1

satisfying $\lim_{\delta_0 \rightarrow 0} \delta_1 = 0$. The surgered surface lies in a Riemannian 3-ball in which we can apply Equation 3. The resulting small volume region in a ball gives a region X in M with $\partial X = F$, since X has canceling boundary (mod 2) on ∂R . Furthermore the volume of X approaches 0 as $\delta_0 \rightarrow 0$. In particular we can arrange that $\text{Volume}(X) < \epsilon_0$ by choosing δ_0 sufficiently small. $\square$

5. HARMONIC AND BOUNDED AREA MAPS

Eells and Sampson showed that a map from a Riemannian surface F with metric h to a negatively curved manifold M is homotopic to a harmonic map [5]. This harmonic map is unique unless its image is a point or a closed geodesic [7], cases that we will not need to consider. It is obtained by deforming an initial map of a surface with a given Riemannian metric (or conformal structure) in the direction of fastest decrease for the energy, called the tension field. The resulting harmonic map depends smoothly on the metrics of both the domain and the image, as shown in Theorem 3.1 of [3]. See also [19]. A homotopy class of maps from a surface to a 3-manifold is *elementary* if it induces a homomorphism of fundamental groups whose image is trivial or cyclic.

Lemma 5.1. *Let $\{g_u : u \in U\}$ be a family of metrics on a closed surface, parametrized by some compact set $U \subset \mathbb{R}^n$ and let M be a closed manifold with negative sectional curvature. Then a smooth family of non-elementary maps $\{f_u : (F, g_u) \rightarrow M\}$ is deformable to a smooth family of harmonic maps $\{h_u : (F, g_u) \rightarrow M\}$.*

Since harmonic maps satisfy a maximal principle, they are more negatively curved than the ambient manifold wherever they are immersed [19]. The Gauss-Bonnet theorem then implies an area bound proportional to the genus. The immersion assumption can be removed by an approximation of a general map by immersions. A detailed argument is found in Theorem 3.2 of [10].

Lemma 5.2. *A harmonic map $f : F \rightarrow M$ from a genus g surface to a hyperbolic 3-manifold M has area bounded above by $4\pi(g - 1)$. If M has sectional curvatures between $-s$ and $-r$, for $0 < r < s$, then its area is bounded above by $4\pi(g - 1)/r$.*

For any Riemannian metric on F , a basic inequality relates the energy and area of a map f [3]:

$$\text{Energy}(f) \geq 2\text{Area}(f).$$

Equality occurs precisely when f is almost conformal, i.e. conformal except possibly at finitely many singular points with zero derivative. In particular, equality holds for an isometric immersion, an immersion of a surface into a Riemannian manifold with the induced metric on the surface.

Lemma 5.3. *A smooth family of immersions $\{f_{u,0} : F \rightarrow M, u \in U\}$ from a surface F to a closed negatively curved 3-manifold M is smoothly deformable to a family of harmonic maps $\{f_{u,1} : (F, g_u) \rightarrow M\}$ through maps $\{f_{u,s}, 0 \leq s \leq 1\}$ satisfying*

$$\text{Area}(f_{u,s}) \leq \text{Area}(f_{u,0}).$$

Proof. We begin by taking the induced metric on the surface F using each of the family of maps $f_{u,0}$. This gives a family of isometric immersions of F , for each of which the energy equals twice the area. The Eells-Sampson process of deforming along the tension field gives an energy decreasing deformation that converges to a harmonic map [5]. Since the energy is non-increasing during this flow, and the area is bounded above by twice the energy, the area of each surface in this flow is bounded above by its initial value. $\square$

6. SWEEPOUTS

Consider a family of maps $k : F \times (-1, 1) \rightarrow M$ such that

$$\lim_{t \rightarrow \pm 1} \text{Area}(k(F, t)) = 0$$

and let F_t denote the surface $k(F, t)$. By Lemma 4.5 we know that the volume of a 3-chain bounded by F_t also approaches zero as $t \rightarrow \pm 1$. It follows that there is a $\beta > 0$ such that one side of F_t has volume less than $\text{Volume}(M)/10$ if $t \in (-1, -1+\beta] \cup [1-\beta, 1)$. In particular $F_{-1+\beta}$ bounds a 3-chain $C_{-1+\beta}$ of volume less than $\text{Volume}(M)/10$ and similarly $F_{1-\beta}$ bounds such a chain $C_{1-\beta}$. Consider the 3-cycle $Z \in H_3(M; \mathbb{Z}_2)$ formed by adding the 3-chains $k(F \times [-1+\beta, 1-\beta])$, $C_{-1+\beta}$ and $C_{1-\beta}$. We say the family of maps k has *degree one* if Z represents the fundamental homology class of M (with $\mathbb{Z}_2$ -coefficients) and in that case we define it to be a *sweepout* of M . Note that changing the value of β continuously changes the volume of Z , while a change in the homology class of Z would cause the volume to change by the volume of M . Thus the choice of β does not affect the question of whether k has degree one.

We digress somewhat to point out that our constructions give a new approach to constructing sweepouts of bounded area. Pitts and Rubinstein showed the existence of an unstable minimal surface in a 3-manifold using a minimax argument. The minimal surface they construct has maximal area in a 1-parameter family of surfaces obtained from a sweepout given by a strongly irreducible Heegaard splitting. Since a genus g minimal surface in a hyperbolic manifold has area less than $4\pi(g-1)$, this implies an area bound for each surface in the sweepout. The existence of a sweepout composed of bounded area surfaces has implications on the geometry and Heegaard genus of a 3-manifold, as noted in Rubinstein [17] and Bachman-Cooper-White [1]. Harmonic maps give an alternate way to obtain the same area bound implied by Pitts-Rubinstein, without finding a minimal surface and without assuming that a Heegaard splitting is strongly irreducible.

Theorem 6.1. *If M is a hyperbolic 3-manifold with a genus g Heegaard splitting then M has a sweepout whose surfaces each has area bounded above by $4\pi(g-1)$.*

Proof. Parametrize each surface of the Heegaard splitting to obtain a smoothly varying family of embedded maps $\{f_t : (F, g_t) \rightarrow M, -1 < t < 1\}$, where g_t is the induced metric on F under the map f_t . By Lemma 5.1, the family of maps is deformable to a smooth sweepout of harmonic maps $\{h_t : (F, g_t) \rightarrow M\}$. Each map h_t has area bounded above by $4\pi(g-1)$, by Lemma 5.2. $\square$

Given a sweepout of a 3-manifold M and two subsets L and R , we can characterize the direction of the sweepout relative to L and R , describing which of L and R is first engulfed. Let β be a constant that satisfies the conditions of Lemma 4.5, so that for $-1 < s \leq -1 + \beta$, F_s bounds a 3-chain of volume less than $\text{Volume}(M)/10$. If $-1 < t < -1 + \beta$ define K_t to be the 3-chain C_t with boundary F_t , given by the smaller volume side of F_t . If $t \geq -1 + \beta$ define K_t to be the 3-chain obtained by adding the 3-chains $k(F \times [-1 + \beta, t])$ and $C_{-1+\beta}$, again with boundary F_t . As t increases on $(-1, 1)$ the volume of K_t changes continuously, and satisfies

$$\lim_{t \rightarrow -1} \text{Volume}(K_t \cap (L \cup R)) = 0$$

and

$$\lim_{t \rightarrow 1} \text{Volume}(K_t \cap (L \cup R)) = \text{Volume}(L \cup R).$$

Let Y denote the set of points $t \in (-1, 1)$ where K_t contains half the volume of $\text{Volume}(L \cup R)$,

$$Y = \{t : \frac{\text{Volume}(K_t \cap (L \cup R))}{\text{Volume}(L \cup R)} = \frac{1}{2}\}.$$

For a generic sweepout this set is finite and odd.

Definition. A point $t \in Y$ is an *L point* if

$$\text{Volume}(K_t \cap L) > \text{Volume}(K_t \cap R)$$

and an *R point* otherwise. A sweepout is an *LR-sweepout* if it has an odd number of *L points* and an *RL-sweepout* if it has an odd number of *R points*.

The Heegaard splitting E_0 gives rise to an *LR-sweepout* that begins with surfaces near a graph at the core of H_L , sweeps out $L \cup R$ with embedded slices, and ends with surfaces that collapse to a graph at the core of H_R . The stabilized Heegaard splitting G_0 gives rise to an *LR-sweepout* by embedded surfaces of genus $2g-1$. A stabilization adds a loop to each of the graphs forming the cores of E_0 , the two loops linking once in M_g , and each crossing the separating surface S . Between the resulting graphs is a product region which can be filled with Heegaard surfaces of the stabilized splitting. See Figure 3, which shows two cores and a surface in G_0 that has been stabilized once. Assuming that G_0 and G_1 are equivalent, composing the resulting

sweepout of G_0 with the diffeomorphisms $\{I_s, 0 \leq s \leq 1\}$ gives a family of genus $2g - 1$ sweepouts connecting G_0 and G_1 .

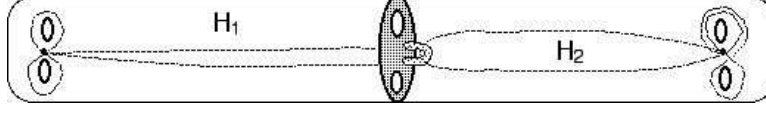


FIGURE 3. Stabilized cores.

Lemma 6.1. *Suppose that E_0 is equivalent to E_1 after $(g-1)$ stabilizations. Then there is a constant A_0 , independent of the number of blocks n , and a family of surfaces*

$$\{f_{s,t} : F \rightarrow M_g, 0 \leq s \leq 1, -1 < t < 1\}$$

satisfying the following conditions:

- (1) F has genus $2g - 1$
- (2) For each s , $\{f_{s,t} : F \rightarrow M_g, -1 < t < 1\}$ is a sweepout.
- (3) $\{f_{0,t}(F), -1 < t < 1\}$ is an LR -sweepout.
- (4) $\{f_{1,t}(F), -1 < t < 1\}$ is an RL -sweepout.
- (5) Each surface in the two sweepouts $f_{0,t}(F)$ and $f_{1,t}(F)$ has area bounded above by A_0 .
- (6) Each surface $\{f_{s,t}(F), 0 \leq s \leq 1, -1 < t < 1\}$ is embedded.

Proof. M_g is formed from the union of a handlebody H_L , n -blocks forming L , n -blocks forming R , and a second handlebody H_R . The geometry of a single block, and of each of H_L, H_R , does not depend on n . Pick a core for each of H_L, H_R and foliate the complement of this core in each handlebody by embedded Heegaard surfaces, connecting the core to the slice that forms the boundary of H_L . Do the same for H_R , and fill L and R with interpolating slices. This gives a foliation of the complement of the two cores in M_g by genus g leaves $\{L_t, -1 < t < 1\}$.

Now stabilize by adding $(g-1)$ loops to each core graph and $(g-1)$ handles to each surface between the two core graphs. By adding thin handles, this can be achieved while increasing the area of any surface by less than a_0 and with the additional $(g-1)$ 1-handles bounding a region of volume less than v_0 , where the constants a_0 and v_0 can be chosen arbitrarily small, independently of n . For our purpose it suffices to pick $v_0 < V_B/10$, where V_B is the volume of a single block, and $a_0 < 1$.

Let F be a surface of genus $2g - 1$ and construct maps $\{f_{0,t} : F \rightarrow M_g, -1 < t < 1\}$ that smoothly parametrize the stabilized surfaces. We will refer to $f_{0,t}(F)$ as $F_{0,t}$. Set the value of the constant A_0 to be the largest area of the surfaces $\{F_{0,t}, -1 < t < 1\}$. Note that our construction gives a value for A_0 that is independent of the number of blocks n in M_g .

The Heegaard splitting $G_0 = (\bar{H}_1, \bar{H}_2, \bar{S})$ obtained by stabilizing E_0 $(g-1)$ times is assumed to be equivalent to the reversed splitting $G_1 =$

$(\bar{H}_2, \bar{H}_1, -\bar{S})$, where $\bar{S} = F_{0,0}$ and $-\bar{S}$ indicates the orientation of $\bar{S}$ has reversed. So an isotopy I_s , $0 \leq s \leq 1$ from the identity map I_0 to a diffeomorphism I_1 carries $(\bar{H}_1, \bar{H}_2, \bar{S})$ to $(\bar{H}_2, \bar{H}_1, -\bar{S})$. Construct the family of surfaces $f_{s,t} : F \rightarrow M_g$ by defining

$$f_{s,t} = I_s \circ f_{0,t} : F \rightarrow M_g$$

and let $F_{s,t}$ denote $f_{s,t}(F)$.

For any constant $\alpha > 0$ we can assume, by stretching out a collar around the invariant surface $F_{0,0}$, that I_1 carries $F_{0,t}$ to $F_{0,-t}$ for each $t \in [-1 + \alpha, 1 - \alpha]$. For α sufficiently small the embedded surfaces

$$\{F_{s,t}; t \in (0, -1 + \alpha] \cup [1 - \alpha, 1)\}$$

lie in small neighborhoods of the images of the core graphs of G_0 under I_s , have area uniformly bounded above by a_0 , and bound submanifolds having volume less than v_0 .

The Heegaard splitting E_0 gives rise to a sweepout of M_g by genus g surfaces $\{L_t, -1 < t < 1\}$ starting near the core of H_L and ending near the core of H_R . This sweepout foliates the complement of the two core graphs. The surface L_t bounds a 3-chain K_t that fills up the side containing $\{\cup L_{t'} : t' < t\}$. K_0 bisects the volume of $L \cup R$ and contains L but not R . The surfaces L_t , when parametrized, give an LR -sweepout with $\text{Volume}(K_0 \cap L) = nV_B$ while $\text{Volume}(K_0 \cap R) = 0$. The surface $F_{0,t}$, obtained by stabilizing L_t , bounds a 3-chain $K_{0,t}$ whose volume of intersection with L and R differs by less than $v_0 < v_B/10$ from that of K_t . Therefore $F_{0,t}$ also gives an LR -sweepout. By the same argument applied to the stabilization of E_1 , we have that $F_{1,t}$ gives an RL -sweepout.

All properties now follow. $\square$

We now show that a path of sweepouts in M_g whose surfaces have area uniformly bounded by a constant A_0 cannot start with an LR -sweepout and end with RL -sweepout, if n is sufficiently large. Define

$$V_L = \text{Volume}(L) = \text{Volume}(R)$$

so that $2V_L = \text{Volume}(L \cup R)$.

Lemma 6.2. *Suppose A_0 is a constant and that M_g is constructed with $2n$ -blocks. Then for n sufficiently large there does not exist a smooth family of maps*

$$\{h_{s,t} : F \rightarrow M_g, 0 \leq s \leq 1, -1 < t < 1\}$$

from a surface F to M_g satisfying the following conditions:

- (1) *F has genus less than $2g$.*
- (2) *For each fixed s , $h_{s,t} : F \rightarrow M_g$ is a sweepout.*
- (3) *Each surface $h_{s,t}(F)$ has area bounded above by A_0 .*
- (4) *$\{h_{0,t}(F), -1 < t < 1\}$ is an LR -sweepout.*
- (5) *$\{h_{1,t}(F), -1 < t < 1\}$ is an RL -sweepout.*

Proof. Suppose such a family exists and let $F_{s,t}$ denote $h_{s,t}(F)$. For each $F_{s,t}$ we construct a 3-chain $K_{s,t}$ with boundary $F_{s,t}$. Pick β sufficiently small to satisfy the conditions of in Lemma 4.5 for each $0 \leq s \leq 1$. Then for each s and each $-1 < t_0 \leq -1 + \beta$ there is a 3-chain C_{s,t_0} with boundary F_{s,t_0} and volume less than $V_L/10$. If $-1 < t < -1 + \beta$ take $K_{s,t}$ to be equal to $C_{s,t}$. If $-1 + \beta < t < -1$ take $K_{s,t}$ to be the sum of $C_{s,-1+\beta}$ and $\bigcup_{-1+\beta \leq t' \leq t} F_{s,t'}$.

The volume of $K_{s,t}$ varies continuously with s and t , giving a continuous function from $[0, 1] \times (-1, 1) \rightarrow \mathbb{R}$. For each fixed s the surfaces $F_{s,t}$ sweep out M_g with degree one, so

$$\lim_{t \rightarrow -1} \text{Volume}(K_{s,t} \cap (L \cup R)) = 0$$

and

$$\lim_{t \rightarrow 1} \text{Volume}(K_{s,t} \cap (L \cup R)) = 2V_L.$$

Set

$$Q = \{(s, t) : \frac{\text{Volume}(K_{s,t} \cap (L \cup R))}{2V_L} = \frac{1}{2}\}.$$

If necessary, perturb the value $1/2$ used to define Q to a regular value of the volume function evaluated on the rectangle. We then have that Q is a 1-manifold and that any path from the edge $t = -1$ to the edge $t = 1$ of the (s, t) rectangle must cross Q . It follows that there is a path contained in Q connecting the edges $s = 0$ and $s = 1$ of the (s, t) rectangle, as in Figure 4.

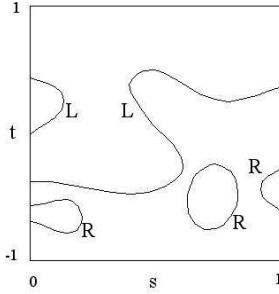


FIGURE 4. Each curve in Q consists entirely of L points or R points.

We now claim that each component of Q consists entirely of L points or entirely of R points. If a component of Q has both L points and R points then there is a point $(s, t) \in Q$ on that component where

$$\text{Volume}(K_{s,t} \cap L) = \text{Volume}(K_{s,t} \cap R)$$

and since these two volumes sum to V_L , we have

$$(4) \quad \frac{\text{Volume}(K_{s,t} \cap L)}{2V_L} = \frac{\text{Volume}(K_{s,t} \cap R)}{2V_L} = \frac{1}{4}$$

The area of $F_{s,t}$ is bounded above by A_0 , so Lemma 4.4 implies that for n sufficiently large, $F_{s,t}$ bounds a 3-chain in L with volume less than $(0.1)V_L$.

Since any two 3-chains in L with the same boundary have volumes in L that sum to a multiple of V_L , we have

$$0 \leq \frac{\text{Volume}(K_{s,t} \cap L)}{V_L} \leq 0.1.$$

or

$$0.9 \leq \frac{\text{Volume}(K_{s,t} \cap L)}{V_L} \leq 1.$$

As a fraction of $2V_L$

$$0 \leq \frac{\text{Volume}(K_{s,t} \cap L)}{2V_L} \leq 0.05$$

or

$$0.45 \leq \frac{\text{Volume}(K_{s,t} \cap L)}{2V_L} \leq 0.5.$$

This contradicts Equation 4. We conclude that components of Q consist either entirely of L or entirely of R points.

A component curve in Q either meets each edge $s = 0$ and $s = 1$ once, meets one edge of the (s, t) rectangle twice, or is disjoint from both. The parity of the number of L points and the number of R points on the two families $h_{0,t}(F)$ and $h_{1,t}(F)$ is the same for each edge $s = 0$ and $s = 1$, and thus either both are LR -sweepouts or both are RL -sweepouts. $\square$

We now show how to replace a family of sweepouts that starts and ends with sweepouts whose surfaces have area less than A_0 , but with no control on the area of surfaces in the intermediate sweepouts, with a homotopic family of sweepouts whose surfaces have area uniformly bounded by A_0 .

Lemma 6.3. *Let $\{f_{s,t,0} : F \rightarrow M_g, 0 \leq s \leq 1, -1 < t < 1\}$ be a family of surface maps such that*

- (1) *The genus of F is less than $2g$.*
- (2) *For each s , $\{f_{s,t,0} : F \rightarrow M_g, -1 < t < 1\}$ is a sweepout.*
- (3) *Each surface in the two sweepouts $f_{0,t,0}(F)$ and $f_{1,t,0}(F)$ has area bounded above by a constant A_0 .*
- (4) *$f_{0,t,0}(F)$, $-1 < t < 1$ is an LR sweepout.*
- (5) *$f_{1,t,0}(F)$, $-1 < t < 1$ is an RL sweepout.*
- (6) *Each surface $\{f_{s,t,0} : F \rightarrow M_g, 0 \leq s \leq 1, -1 < t < 1\}$ is immersed.*

Then if n is sufficiently large, there is a family of maps

$$\{f_{s,t,u} : F \rightarrow M_g, 0 \leq s \leq 1, -1 \leq t \leq 1, 0 \leq u \leq 1\}$$

such that

- (1) *For each fixed s, u ; $\{f_{s,t,u} : F \rightarrow M_g, -1 \leq t \leq 1\}$ is a sweepout.*
- (2) *Each surface $f_{s,t,1}(F)$ is harmonic, with area at most $8\pi(2g - 1)$.*
- (3) *$f_{0,t,1}(F)$, $-1 < t < 1$ is an LR sweepout.*
- (4) *$f_{1,t,1}(F)$, $-1 < t < 1$ is an RL sweepout.*

Proof. We define a family of Riemannian metrics $h_{s,t}$ on F for each $0 \leq s \leq 1$, $-1 < t < 1$ by taking the induced metric pulled back from M_g by the immersion $f_{s,t,0}$. Then each map $f_{s,t,0}$ is conformal and has energy is equal to twice its area. We flow the family $f_{s,t,0}$ to a family of harmonic maps $f_{s,t,1}$ as in Lemma 5.3, with the area of each harmonic surface $f_{s,t,1}(F)$ less than A_0 . Each surface in the sweepout $f_{0,t,0}(F)$ has area bounded above by A_0 , and this area bound is maintained for each surface $\{f_{0,t,u}(F), 0 \leq u \leq 1\}$ in the homotopy to harmonic maps, as in Lemma 5.2. Similarly the homotopy to harmonic maps of the sweepout $f_{1,t,0}(F)$ given by $\{f_{1,t,u}(F), 0 \leq u \leq 1\}$ has surfaces whose area is uniformly bounded above by A_0 . Thus if n is sufficiently large, the sweepout $\{f_{0,t,1}(F)\}$ is an LR sweepout, by Lemma 6.2 and similarly $\{f_{1,t,1}(F)\}$ is an RL sweepout. $\square$

7. INVERTING HEEGAARD SURFACES

We now prove the main result:

Proof of Theorem 1.1. We begin with the two genus g Heegaard foliations, E_0 and E_1 of M_g . We will show that these splittings are not k -stably equivalent for $k < g$ if n is sufficiently large. It suffices to take $k = g - 1$.

Assume to the contrary that for all n , G_0 and G_1 are equivalent after $g - 1$ stabilizations. For n large, Lemma 6.1 implies there is a family of sweepouts

$$\{f_{s,t,0} : F \rightarrow M_g, 0 \leq s \leq 1, -1 < t < 1\}$$

of genus $2g - 1$ with $\{f_{0,t,0}(F), -1 < t < 1\}$ an LR -sweepout, $\{f_{1,t,0}(F), -1 < t < 1\}$ an RL -sweepout and that each surface in the two sweepouts $f_{0,t,0}(F)$ and $f_{1,t,0}(F)$ has area bounded above by A_0 . Lemma 6.3 implies that this family can be deformed to a new family of sweepouts

$$\{f_{s,t,1} : F \rightarrow M_g, 0 \leq s \leq 1, -1 < t < 1\}$$

in which $\{f_{0,t,1}(F), -1 < t < 1\}$ is an LR -sweepout, $\{f_{1,t,1}(F), -1 < t < 1\}$ is an RL -sweepout and each surface $f_{s,t,1}(F)$ has area bounded above by A_0 . This is a path of sweepouts all of whose surfaces have area less than A_0 that connects an LR -sweepout to a RL -sweepout. Lemma 6.2 states that no such path of sweepouts can exist, contradicting the assumption that fewer than g stabilizations can make E_0 and E_1 equivalent. $\square$

8. A HYPERBOLIC EXAMPLE

The Riemannian manifolds M_g used in our construction were negatively curved, but not hyperbolic. We now show how to construct a family of hyperbolic manifolds that give somewhat weaker lower bounds on the number of stabilizations required to make two genus- g splittings equivalent. Note that block manifolds cannot be isometrically embedded into a hyperbolic 3-manifold in which their slices are Heegaard surfaces, because the fibers of a surface bundle lift to planes in the universal cover. However there is an isometric embedding of block manifolds into a hyperbolic manifold in

which slices are separating incompressible surfaces. These surfaces become Heegaard surfaces after two stabilizations.

Let N_0 be a hyperbolic 3-manifold that is a union of two I -bundles over a non-orientable surface, glued along their common genus k boundary surface, where k is an integer greater than one. Such hyperbolic manifolds are double covered by a hyperbolic surface bundle over S^1 . Some explicit examples can be found in [16]. The boundary surface of each I -bundle has a neighborhood isometric to a neighborhood of a fiber in its double cover. We can cut open along this fiber and insert a block manifold with arbitrarily many blocks to obtain a hyperbolic manifold N , still constructed as a union of two I -bundles. We let S denote a surface separating the two I -bundles in the center of the block manifold.

Removing a neighborhood of an interval fiber from an I -bundle with a genus k boundary surface results in a handlebody of genus $k + 1$. Thus N has a genus $g = k + 2$ Heegaard surface S' obtained by adding a 1-handle to each side of S , with the core of each 1-handle an interval in each I -bundle. The two orderings of these handlebodies give rise to two Heegaard splittings of N , and to corresponding sweepouts that fill the two I -bundles in opposite order. The arguments used on M_g in Section 7 now apply to show that N has two genus g splittings that require no less than $g - 4$ stabilizations to become equivalent.

REFERENCES

- [1] D. Bachman, D. Cooper and M. White, *Large embedded balls and Heegaard genus in negative curvature*, Algebraic & Geometric Topology 4 (2004) 31-47.
- [2] I. Chavel, *Isoperimetric inequalities*, Cambridge Univ. Press, Cambridge, UK, 2001,
- [3] J. Eells and L. Lemaire, *Deformations of metrics and associated harmonic maps*, Proc. Indian Acad. Sci. 90 (1) (1981), 33-45.
- [4] J. Eells, Jr. and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 10 (1978), 1-68.
- [5] J. Eells and Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109-160.
- [6] H. Federer, *Geometric measure theory*. Springer-Verlag, Berlin, Heidelberg, New York, 1969.
- [7] P. Hartman, *On homotopic harmonic maps*, Canad. J. Math. 19 (1967), 673- 687.
- [8] J. Hass, *Minimal surfaces and the topology of 3-manifolds*, Global theory of minimal surfaces, Clay Math. Proc., 2, Amer. Math. Soc., Providence, RI, (2005) 705724.
- [9] R. Kirby, Problems in low-dimensional topology, preprint, <http://math.berkeley.edu/~kirby>.
- [10] Y. Minsky, *Harmonic Maps into Hyperbolic 3-Manifolds*, Transactions of the American Mathematical Society, 332 (1992), 607-632.
- [11] Y. Minsky, *Harmonic maps, length and energy in Teichmüller space*, J. Differential Geom. 35 (1992), 151-217.
- [12] H. Namazi and J. Souto, *Heegaard splittings and pseudo-Anosov maps*, preprint, <http://www.math.princeton.edu/~hossein/contents/pseudo8.pdf>

- [13] W. H. Meeks III, Lectures on Plateau's problem, IMPA, Rio de Janeiro, Brazil, 1978
- [14] E. Moise, Geometric Topology in Dimensions 2 and 3, Springer-Verlag, (1977).
- [15] J. Pitts and J. H. Rubinstein, *Applications of minimax to minimal surfaces and the topology of 3-manifolds*, , Proc. Centre Math. Applic., Australian National University 12, (1987), 137-170.
- [16] A. Reid *A non-Haken hyperbolic 3-manifold covered by a surface bundle*, Pacific J. Math. 167 (1995), 163182.
- [17] J. H. Rubinstein, *Minimal surfaces in geometric 3-manifolds*, Global theory of minimal surfaces, Clay Mathematics Proceedings Volume 2, ed D. Hoffman, Amer. Math. Soc. (2005), 725-746.
- [18] J.H. Rubinstein and M. Scharlemann, *Comparing Heegaard splittings of non-Haken 3-manifolds* Topology 35 (1996), 1005-1026.
- [19] J.H. Sampson *Some properties and applications of harmonic mappings*, Annales scientifiques de l'cole Normale Suprieure Sr. 4, 11 no. 2 (1978), p. 211-228.
- [20] M Scharlemann, *Heegaard splittings of compact 3-manifolds*, Handbook of geometric topology, ed by R. Daverman and R. Sherr, 921-953, North-Holland, Amsterdam, 2002.
- [21] E. Sedgwick, *An infinite collection of Heegaard splittings that are equivalent after one stabilization*, Math. Ann. 308 (1997), 65-72.
- [22] J. Schultens *The stabilization problem for Heegaard fibered spaces*, Topology and its Applications 73 (1996) 133-I 39.
- [23] J. Singer, *Three-Dimensional manifolds and their Heegaard diagrams*, Transactions of the American Mathematical Society, Vol. 35, No. 1. (Jan., 1933), pp. 88-111.
- [24] M. Wolf, *Measured Foliations and Harmonic Maps of Surfaces*, J. Differential Geom., 49 (1998): 437-467.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS CALIFORNIA 95616

E-mail address: `hass@math.ucdavis.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS CALIFORNIA 95616

E-mail address: `thompson@math.ucdavis.edu`

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853

E-mail address: `wpt@math.cornell.edu`