MINIMAL SURFACES AND THE TOPOLOGY OF 3-MANIFOLDS

JOEL HASS

ABSTRACT. This article, based on three lectures given at MSRI in July 2001, discusses old and new results relating minimal surface theory and the topology of 3-manifolds. The goal is to give a picture of how minimal surfaces have become an important tool in the study of 3-manifolds.

1. INTRODUCTION

In this paper we discuss some old and new results relating minimal surface theory and the topology of 3-manifolds. We have selected only a few of the many wonderful results in this area, but hopefully enough to give the flavor of its ideas. For the sake of exposition we keep the discussion somewhat informal, and don't try to give completely general statements. Detailed proofs and general statements can be found in the cited references. The article is based on three lectures given in July, 2001 at the Clay Mathematics Institute 2001 Summer Workshop on the Global Theory of Minimal Surfaces.

Throughout this paper, F will refer to a compact surface and M to a closed 3-dimensional Riemannian manifold, one that is compact and has no boundary. All maps and spaces will be assumed C^∞ unless otherwise noted.

2. EXISTENCE

Establishing the existence of least area surfaces was an important problem in the 19th and 20th centuries. The modern formulation is usually ascribed to the Belgian physicist Plateau [53], who studied soap films experimentally and instigated the exploration of their mathematical properties. The Plateau problem had various formulations, depending on how close to the soap bubble model one wanted to stay. A common version was stated as follows:

Plateau's Problem: Let Γ be a rectifiable, simple closed curve in $\mathbb{R}^3$. Show that Γ bounds a smoothly immersed spanning disk of least area.

To formulate this more precisely, define $\mathcal{F}$ to be the family of piecewise smooth maps $f : D^2 \to R^3$ such that $f|_{\partial D^2}$ is a homeomorphism from ∂D^2 to Γ. Let $\mathcal{I} = inf\{Area(f) : f \in \mathcal{F}\}$. The problem is to find a map in $\mathcal{F}$ with area $\mathcal{I}$.

This statement allows for solutions that are immersed disks ,possibly self-intersecting, and thus is not a faithful model of the behavior of soap films in space. However

Date: June 18, 2003.
1991 *Mathematics Subject Classification.* Primary 57N10, 53A10; Secondary 68Q25.
Key words and phrases. Minimal surface, 3-manifold, complexity of algorithms, normal surface, area.
Partially supported by NSF grant DMS-0072348 and by a grant from the Clay Mathematical Institute.

this formulation is well suited to the study of 3-manifolds, as well as to solution by analytic methods.

There were many partial results, and a complete solution in $\mathbb{R}^n$ was obtained by Douglas and Rado around 1930. See [56] for a historical summary. The two techniques are rather different. Rado developed a method based on minimizing energy rather than area. The energy of a map $f(u_1, u_2)$ from the disk to $\mathbb{R}^k$ is given by the Dirichlet integral,

$$\int\int_D \sum_{k=1}^{n} (\frac{\partial f_k}{\partial u_1})^2 + (\frac{\partial f_k}{\partial u_2})^2 \ du_1 \ du_2 \ .$$

Minimizing with respect to fixed boundary values parameterizing Γ leads to a harmonic map. Minimizing energy among all boundary values monotonically parameterizing Γ gives a least area map. The resulting map is conformal, except possibly at a finite number of singularities, called *branch points*. See [6], [45] for a more extensive discussion. Douglas took a different approach, and minimized what is now called the Douglas Integral [8]. This proved harder to extend to general metrics and topologies than the Dirichlet integral, though Douglas achieved some success in this direction.

Morrey extended the existence results for minimal disks to null-homotopic curves in Riemannian manifolds that are homogeneously regular [44].

Theorem 2.1 (Douglas, Rado, Morrey). *Let Γ be a rectifiable, null-homotopic, simple curve in a compact Riemannian 3-manifold. Then Γ bounds a disk of least area.*

The assumption that the curve is embedded can be removed [22].

There is an alternative approach to existence results based on Geometric Measure Theory, completed for surfaces in 3-dimensional manifolds by Federer and Fleming [11], [10]. With further work of Hardt and Simon [19] on boundary regularity, this gave a solution of a version of Plateau's problem more closely modeling a physical soap film. We will concentrate on the previous formulation, which is well suited to investigation of problems in low dimensional topology.

We fix some terminology. An immersion $f : F \to M$ of a compact surface to a Riemannian manifold is *minimal* if it has zero mean curvature. More generally, we consider minimal surfaces that fail to be immersions at isolated branch points. A map of a surface is *least area* if it has no larger area than any homotopic map. A map of a surface is *area minimizing* if it is homologically minimizing, so that the area of its image is no larger than that of any surface in the same homology class. If the surface is non-compact, we can use the same definitions, but two competing surfaces are required to agree off a compact subset.

3. Branch Points

Minimal surfaces in $\mathbb{R}^n$ are immersed except at isolated singular points, where their structure can be understood through the properties of harmonic functions, which form the coordinate functions. These surfaces, which include the solutions to the Plateau problem in $\mathbb{R}^3$ produced by Douglas and Rado, are "conformal branched immersions".

For least area surfaces in $\mathbb{R}^3$ it was unknown whether singularities actually do occur. Osserman showed that branch points do not exist in the interior of least area surfaces. He established that least area surfaces have immersed images, using an argument similar to one used by Dehn in his study of generic maps of surfaces into 3-manifolds [46], [7]. Osserman's work was supplemented by further contributions of Gulliver and Alt, who established also that no "false" branch points can occur in a least area surface [14], [3]. These are singularities of a map due to a bad parameterization rather then a singularity in the image, These methods apply equally well in a general Riemannian 3-manifold.

Theorem 3.1. *A least area surface in a Riemannian 3-manifold is immersed in its interior.*

Idea of the proof: Suppose a least area surface has an interior branch point. The local self-intersections of the surface near a branched point can be analyzed, and reveal that there is a curve of double points leading into the branched point. Cutting and pasting along that double curve near the branch point leads to a homotopic surface of the same genus and area. The resulting surface has a "fold" along the double curve, a curve of points where it fails to be smoothly immersed. But minimal surfaces have only isolated singularities, and so there is a smaller surface in the homotopy class, contradicting the least area hypothesis. □

The possible existence of branch points at the boundary of a least area surface remains open, though important special cases have been shown to be free of such boundary singularities. The structure of a boundary branch point is described in [16]. In [15], Gulliver constructs a minimal surface with a boundary branch point which cannot be eliminated using the methods that were used for interior branch points. It is known that boundary branch points do not exist in many settings, such as when a curve lies on the boundary of its convex hull, or is analytic [16]. To the extent that this question remains open, theversion of the Plateau problem stated above is not completely settled.

4. Embeddedness

Osserman asked the following question: If Γ is a simple closed curve on the 2-sphere and D is a least area disk spanning Γ, is D embedded? This stimulated a series of important results. Special cases followed from work of Rado [56], Gulliver-Spruck [17], Tomi-Tromba [67] and Almgren-Simon [2]. A general positive solution was obtained by Meeks and Yau, who proved the following very general result [39].

Theorem 4.1 (Meeks-Yau Dehn's Lemma). *If D is a least area disk whose boundary is a simple closed curve on the boundary of a Riemannian 3-manifold with mean convex boundary, then D is embedded.*

Mean convex is a generalization of convex that allows for manifolds whose boundaries have positive (or non-negative) mean curvature. Mean convex is a sufficient condition to allow construction of a minimizing sequence whose elements don't bump against the boundary, a key step in establishing existence of area minimizers.

The Meeks-Yau Dehn's Lemma had profound implications in 3-manifold topology. It is not obvious that ∂D bounds any embedded disk, least area or not. That it does is the content of the somewhat notorious Dehn's Lemma, first stated by Dehn in 1920 [7]. Dehn's argument contained a gap, and the result was finally proved in 1957 by Papakyriakopolous [47]. The existence of a least area embedded disk leads to a stronger, equivariant version of Dehn's Lemma. In this setting we have a finite group G acting on a 3-manifold M. By averaging a metric, we can assume that the group acts by isometries. A curve or surface in M is *equivariant* if its translates under the group are either disjoint from it or coincide with it.

Theorem 4.2 (Meeks-Yau Equivariant Dehn's Lemma). *Suppose M is a Riemannian 3-manifold with mean convex boundary and G is a finite group acting as isometries on M. Suppose also that Γ is a null-homotopic curve on ∂M that is equivariant. If D is a least area disk with boundary Γ then D is embedded and equivariant.*

As an application, we show how to classify $\mathbb{Z}_3$ actions on the solid torus

$$S^1 \times D^2 = \{(\theta_1, \theta_2, r) : 0 \leq \theta_1 < 2\pi, 0 \leq \theta_2 < 2\pi, 0 \leq r \leq 1\}.$$

An action is *standard* if it is conjugate to an action where a generator τ of $\mathbb{Z}_3$ rotates one or both circles of the torus. That is $\tau(\theta_1, \theta_2, r) = (\omega\theta_1, \mu\theta_2, r)$ where ω, μ are cube roots of unity.

Theorem 4.3. *Suppose $\mathbb{Z}_3$ acts on $S^1 \times D^2$. Then the action is standard.*

Sketch of Proof: Take any metric on $S^1 \times D^2$. Average it over the group action to get a new metric on which the group acts as isometries. Let β be a geodesic on the boundary torus that is shortest among all non-trivial curves on the torus that bound disks in the solid torus. The three translates β, $\tau(\beta)$, $\tau^2(\beta)$ are embedded and either disjoint or equal, as otherwise one could find a homotopic curve shorter than β. Take a least area disk D with boundary β. By the Meeks-Yau Equivariant Dehn's Lemma, the three translates β, $\tau(D)$, $\tau^2(D)$ are embedded and either disjoint or equal. Together they split the solid torus into either one or three balls. If three, the three balls are rotated one to another by τ. If one, the action of the group takes this ball to itself. Actions of $\mathbb{Z}_3$ on the 3-ball were shown to be standard in the proof of the Smith Conjecture, by a combination of results including another application of the Meeks-Yau Dehn's Lemma [43]. It follows in each case that the action on the solid torus is standard. □

Much more general groups are shown to act standardly on many 3-manifolds by similar techniques [37].

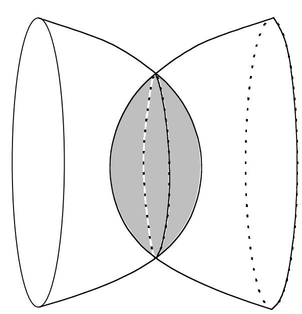

FIGURE 1. A product region cannot occur in a least area immersion.

The basic idea in proving the embeddedness results such as Thorem 4.1 is to find "parallel" subsurfaces of an immersed disk to exchange, by cutting and pasting along suitable double curves. The study of such double curves is a notoriously difficult area in 3-manifold topology. If one is fortunate enough to have a "product region" as in Figure 1, then one can interchange the two pieces, smooth off, and reduce area without changing topology or homotopy class. In Figure 1 the product region is a ball, homeomorphic to a disk $\times\ I$. It is not clear that one can always find such regions in an immersed surface with excess intersections, and in fact one sometimes cannot. The difficulty arises from immersions with intersecting double curves, which lead to triple points where three sheets of the surface cross. These triple points are difficult to analyze. A covering space technique introduced by Papakyriakopolous allows us to avoid the problem. A sequence of covering spaces of neighborhoods of the immersed disk is constructed that unwrap the immersed disk until one of two things happens. Either a cover is constructed in which a lift to that cover does indeed contain a product piece, or a smaller area disk is constructed in some cover by taking a component of the boundary of a slight thickening of the immersed disk. In either case, a smaller disk can be constructed in a cover, and projected down to give a smaller disk in the original space. By constructing a smaller area surface with the same boundary, one arrives a contradiction if the original disk was not embedded.

5. Closed surfaces

Closed minimal surfaces, compact and without boundary are not present in $\mathbb{R}^3$, but they do play a key role in the study of 3-manifolds. Since they don't have boundary, we need to impose some topological condition to stop a minimizing surface from collapsing to a point. We first look at minimal spheres, which are treated somewhat differently then higher genus surfaces. We consider a closed Riemannian 3-manifold that has non-zero second homotopy group $\pi_2(M)$. An existence theorem in this context was established by Sacks and Uhlenbeck. Let $\mathcal{F}$ be the family of smooth maps $f : S^2 \to M$ such that f is not homotopic to a point, or equivalently, f represents a non-trivial element of $\pi_2(M)$. (We are suppressing mention of a base point for simplicity). Set $\mathcal{I} = inf\{Area(f) : f \in \mathcal{F}\}$.

Sacks and Uhlenbeck showed that while a particular element of the second homotopy group may or may not contain a least area representative, there exists an element of smallest area among *all* non-trivial 2-spheres.

Theorem 5.1 (Sacks-Uhlenbeck)**.** *If $\pi_2(M)$ is nontrivial then there is map in $\mathcal{F}$ having area $\mathcal{I}$.*

The question of whether *every* representative of $\pi_2(M)$ has a minimal representative, asked by Eells and Lemaire [9, pp. 417], remains open. This is equivalent to asking whether there is a harmonic representative in every homotopy class, since the notion of harmonic map and minimal surface coincide for a 2-sphere domain.

Open Problem 1. *Let M be a closed Riemannian manifold and $f : S^2 \to M$ a map that is homotopically non-trivial. Does f have a minimal representative?*

Probably the answer is no. One approach to constructing a Riemannian 3-manifold with a homotopy class containing no minimal representative is to take a 3-manifold and remove two small balls far away from each other. Then glue in a non-trivial 3-manifold, such as a punctured 3-torus, to replace each of the removed balls. Consider a minimal 2-sphere representing the homotopy class given by the sum in $\pi_2(M)$ of the boundaries of the two removed balls. It seems unlikely that a minimal 2-sphere could span the gap between the balls if the metric is arranged so that they are sufficiently far away from each other. This can be made rigorous for manifolds with convex boundary. For example, construct a Riemannian manifold by taking a unit ball in $\mathbb{R}^3$, removing radius ϵ balls around $(-1/2, 0, 0)$ and $(1/2, 0, 0)$, with ϵ small, and changing the metric near these balls to make a convex boundary. Then no minimal 2-sphere can enclose both the balls. A catenoid of appropriate size would be tangent to such a 2-sphere while not crossing it, contradicting the maximum principle for minimal surfaces [54]. In a closed manifold, an example of a homotopy class of 2-spheres with no minimal representative has not yet been found.

There is an important topological result about embedding spheres in 3-manifolds, the Sphere Theorem, also due to Papakyriakopolous [47]. It states that if $\pi_2(M)$ is nontrivial then there is an embedded, homotopically non-trivial 2-sphere in M, and its proof is similar in flavor to that of Dehn's Lemma, though somewhat more involved. Meeks and Yau extended the Sacks-Uhlenbeck Theorem, and proved least area and equivariant versions of the Sphere Theorem [39].

Theorem 5.2 (Meeks-Yau Sphere Theorem). *If M is a Riemannian manifold which is closed or compact with mean convex boundary, and if $\pi_2(M)$ is nontrivial, then there is a generating set of $\pi_2(M)$, as a $\pi_1(M)$-module, consisting of minimal 2-spheres. Each 2-sphere is least area in its homotopy class, is either embedded or double covers an embedded RP^2, and any pair of 2-spheres is disjoint.*

Given a finite group acting as isometries on a 3-manifold, the Meeks-Yau sphere theorem has an equivariant version, similar to the equivariant Dehn's Lemma. This led to furhter new results on classifying group actions on 3-manifolds, including groups acting on $\mathbb{R}^3$ [38].

Another important result that had implications to 3-manifold topology was stated by W. Meeks, L. Simon and S.T. Yau [41]. Its proof is based more on Geometric Measure Theory then the other results we've discussed.

Theorem 5.3. *Let F be an embedded surface in a Riemannian 3-manifold M. Then after a series of compressions, isotopies, and collapsings of the boundaries of I-bundles to their cores, F can be realized as a union (possibly empty) of disjoint embedded minimal surfaces.*

The phenomenon of collapsing I-bundles can be seen, one dimension down, in the Mobius band, where a curve isotopic to the boundary can be homotoped to double cover a central core curve. With an appropriate Riemannian metric, the shortest curve cna be forced to double cover the core. In three dimensions this can occur even when the surface and the 3-manifold are both orientable.

A 3-manifold is called *irreducible* if any 2-sphere is the boundary of a 3-ball, so that M contains no non-trivial embedded 2-spheres. Cutting along such non-trivial 2-spheres is an important first step in the (not yet completed) classification of 3-manifolds. Theorem 5.3 has the following important consequence, obtained by minimizing among all spheres not bounding balls [41].

Theorem 5.4 (Meeks-Simon-Yau). *Let M be an irreducible orientable 3-manifold M and $\tilde{M}$ a covering space of M. Then $\tilde{M}$ is irreducible.*

We now turn to higher genus surfaces. A common approach in trying to understand 3-manifolds is to cut them open along surfaces into simpler building blocks, and to understand the ways that these pieces are recombined to form the original manifold. For the process to be useful, the cutting surfaces should reflect the global nature of the 3-manifold, as otherwise the building blocks can be more complicated than the original manifold. Experience shows that it is counterproductive to cut open along a surface with lots of knotted tubes, or with complicated self-intersections. Two classes of surfaces are commonly used to cut up 3-manifolds. *Incompressible surfaces* are embedded and contain no trivial tubes or handles, while *Heegaard surfaces* cut the 3-manifold into two handlebodies. Even with these surfaces, careful choices must be made for the cutting procedure to be useful. See the article of Rubinstein in these Proceedings for a discussion of Heegaard minimal surfaces [58]. We look here at incompressible (also called π_1-*injective*) surfaces, defined to be surfaces (other than 2-spheres) whose fundamental groups inject into the fundamental group of M. The basic existence theorem for this class of surfaces was established by Schoen-Yau [61].

Theorem 5.5 (Schoen-Yau). *If a closed Riemannian manifold M contains an incompressible surface of genus $g \geq 1$, then a smallest such surface exists and is immersed.*

The proof of the Schoen-Yau theorem actually shows that a least area surface exists in the homotopy class of each incompressible surface, if $\pi_2(M) = 1$. An embedding theorem for maps of incompressible surfaces was proven by Freedman-Hass-Scott [12].

Theorem 5.6. *The least area surface homotopic to an embedded, incompressible, orientable surface in a closed, orientable, irreducible Riemannian manifold is either embedded or double covers an embedded one-sided surface.*

More generally, a collection of such least area surfaces is shown to minimize its self-intersections [12]. These intersection properties were not previously known even for shortest geodesics on a surface with a general metric. Applications to 3-manifolds of this result include a theorem of Peter Scott about the topological rigidity of Seifert Fiber spaces [63]. This class of 3-manifolds includes manifolds that contain immersed, but not embedded, incompressible tori. Waldhausen had proved in [69] that a 3-manifold M containing an incompressible embedded surface is *topologically rigid*, i.e. any irreducible 3-manifold with isomorphic fundamental group is homeomorphic to M. Waldhausen's methods failed for Seifert Fiber spaces that contain immersed, but not embedded, incompressible tori, but Scott was able to extend the result to these manifolds through the use of least area immersed tori.

Another important theorem of Waldhausen states that if an irreducible manifold contains an embedded, incompressible surface then the universal cover of the manifold is homeomorphic to $\mathbb{R}^3$. This was extended by Hass-Rubinstein-Scott using least area techniques, removing the embeddedness condition from Waldhausen's assumptions [26].

Theorem 5.7. *The universal cover of an irreducible manifold containing an immersed, incompressible surface is homeomorphic to $\mathbb{R}^3$.*

6. Some open problems on minimal surfaces in 3-manifolds

We discuss here a small number of open problems relating minimal surface theory and 3-dimensional topology. More extensive lists of problems can be found in [36] and [71].

The famous Poincare Conjecture asserts that a closed, simply connected 3-manifold is homeomorphic to the 3-sphere. This is equivalent (by removing a ball) to the claim that a compact 3-manifold that is simply connected and has 2-sphere boundary is homeomorphic to a 3-ball. A minimal surface approach to the Poincare conjecture was suggested in the 1980's by M. Freedman and S.T. Yau. We will sketch this plan, which has not been completed to date.

Suppose E is a *fake 3-ball*, a simply connected 3-manifold with boundary a 2-sphere that is not homeomorphic to a ball. Take a Riemannian metric on E in which the boundary is convex and in which there are only finitely many disjoint embedded minimal 2-spheres. Such a metric exists by a theorem of White [70]. The boundary 2-sphere cannot be isotoped to a point in E, so it is isotopic to a stable minimal 2-sphere by Theorem 5.3. Take a stable minimal 2-sphere not bounding a ball that does not contain any other such 2-sphere inside it, and let N be the 3-manifold bounded by this 2-sphere. Then N is also a fake ball with smooth, mean convex boundary. Now consider a curve γ on ∂N. The two disks on ∂N bounded by γ are each stable minimal disks. We expect to be able to find an unstable, immersed minimal disk in the interior of N that lies on a family of disks connecting these two stable disks.

Open Problem 2. *Among all such curves γ, is the smallest area unstable minimal disk embedded?*

This looks intriguingly similar to the Meeks-Yau Dehn's Lemma, which states that a least area minimal disk exists and is embedded for any such γ.

If the answer is yes, then N is divided into two pieces, with common boundary the embedded minimal disk. Each of these regions is simply connected, but at least one is not a ball, since N is not a ball and the union of by two balls along a disk is also a ball. Let P be a component that is a fake ball. The boundary of P is mean convex, so we can apply the Meeks-Simon-Yau Theorem to find a stable minimal 2-sphere inside P, contradicting our hypothesis that ∂N was innermost among such 2-spheres. The Poincare conjecture would follow.

We now turn to a collection of questions concerning one-sided surfaces. A surface immersed in a 3-manifold is *one-sided* if its normal bundle is non-trivial. Otherwise it is *two-sided.* A map of a surface $f : F \to M$ is π_1*-injective* if it induces an injective homomorphism from $\pi_1(F)$ to $\pi_1(M)$.

Conjecture 3. *Let $f : F \to M$ be a one-sided, least area, π_1-injective map of a closed surface F to an irreducible Riemannian 3-manifold M. If f is homotopic to an embedding then f is embedded.*

This is known for two-sided surfaces [12] (up to the phenomenon of double covering a one-sided embedding) and for both one-sided and two-sided geodesics on surfaces [25]. It follows from Theorem 5.3 that there is a one-sided surface homotopic to f that is embedded or a 2-1 cover of an embedding and that minimizes area among all such surfaces. This surface minimizes area in the isotopy class of f (allowing maps that factor through double coverings to be included in this class). However it is not known whether this surface minimizes area in its homotopy class. There exists a least area map in the homotopy class of f by [61], and conceivably in some manifolds there is a least area map whose image crosses itself that is smaller than any embedding.

Closely related is the following:

Conjecture 4. *Let $f : F \to M$ be a one-sided map from a closed surface F to a Riemannian manifold M that induces an isomorphism of fundamental groups. If f has smallest area among such maps, then f is embedded.*

Here we assume that f induces an isomorphism of fundamental groups, rather than just an π_1-injective homomorphism. However we do not assume that f is homotopic to an embedding, nor that M is irreducible. Existence of a least area map is known as is the existence of a least area embedding among all embeddings [61], [41]. Conjecture 4 has some interesting implications, including the topological Conjecture 5. Conjecture 4 implies that if a 3-manifold has fundamental group $\mathbb{Z}_2$ then there is a least area embedded RP^2 inside it. If the manifold is also irreducible, then the boundary of a neighborhood of this RP^2 is a 2-sphere that bounds a 3-ball, necessarily on the other side. The union of a neighborhood of the embedded RP^2 and the 3-ball is homeomorphic to RP^3, which implies a solution to the following open problem in 3-manifold theory.

Conjecture 5 (No fake irreducible RP^3's). *If M is an irreducible 3- manifold homotopy equivalent to RP^3 then M is homeomorphic to RP^3.*

Conjecture 5 follows from Conjecture 4, since a minimal RP^2 would be embedded, and irreducibility would then imply that M is homeomorphic to RP^3.

A related topological conjecture is:

Conjecture 6. *Every homotopy 3-sphere admits a free involution.*

This is certainly true for the 3-sphere, and thus follows from the Poincare conjecture. Together with Conjecture 5, Conjecture 6 would imply the Poincare Conjecture in dimension 3.

Closely related is the following conjecture.

Conjecture 7. *Let M be irreducible and homotopy equivalent to RP^3, with the boundary of M consisting of two copies of RP^2. Let A be a least area annulus whose boundary curves are generators of the fundamental group of each of the two boundary RP^2's. Then A is embedded.*

The above manifold gives an h-cobordism of RP^2 to itself. A corollary of Conjecture 7 is the following topological conjecture.

Conjecture 8. *There are no fake h-cobordisms of RP^2. Equivalently, there is no fake irreducible $RP^2 \times I$.*

The following case of the above type of question is perhaps the simplest, but still seems quite hard.

Conjecture 9 (Meeks). *Let M be a Riemannian manifold diffeomorphic to $S^1 \times D^2$, the solid torus, with mean convex boundary. Let F be a least area Mobius band bounding an embedded (2,1) curve on the boundary torus. Then F is embedded.*

Even the special cases where M is a submanifold of $\mathbb{R}^3$, where M has the product metric, or where M is the standard solid torus of revolution are unknown. Some special cases that can be solved are when the boundary curve is invariant under a circle action or when the total curvature of the boundary curve is at most 6π. Two other special cases of Conjecture 4 are the following.

Conjecture 10. *Let $f : RP^2 \to RP^3$ be a least area π_1-injective map, where RP^3 is given an arbitrary Riemannian metric. Then f is an embedding.*

This conjecture has a generalization to more complicated types of surfaces. A generalized projective plane is a disk except for a curve of singularities along its boundary where locally p-surfaces meet, see [5]. It is not known in what generality least area singular surfaces of this type exist.

Open Problem 11. *Let S be a least area, π_1-injective generalized projective plane in a homotopy lens space. Then S is embedded.*

Also of interest are non-π_1-injective incompressible surfaces. These embedded surfaces contain immersed curves that bound disks with embedded interiors in their complement, but no embedded curves of this type. They are always one-sided. Examples occur in some lens spaces and many other 3-manifolds, and are of interest in the study of 3-manifolds containing no incompressible surfaces.

Open Problem 12. *Let F be a non-π_1-injective incompressible surface in a Riemannian 3-manifold M. Is a least area surface homotopic to F embedded?*

7. Minimal foliations

Given a foliation of a 3-manifold M, under certain conditions it is possible to find a Riemannian metric on M in which every leaf is a minimal surface. This leads to interesting results in both 3-manifolds and minimal surface theory [21].

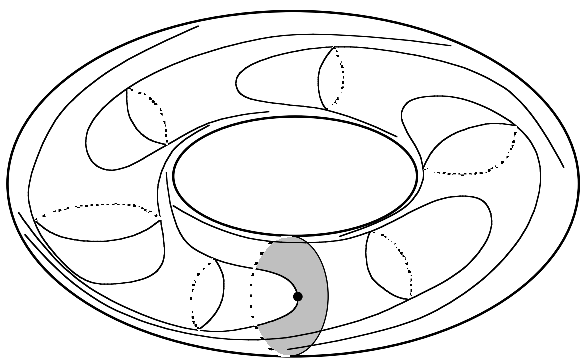

Figure 2. The existence of both minimal leaves and a minimal disk violates the maximum principle.

One obstruction to the existence of such a metric is the existence of Reeb components. These are solid tori foliated as in Figure 2. Suppose that each Reeb leaf is a minimal surface in some metric, including the boundary torus. Consider a least area compressing meridian disk for the solid torus. Such a disk can be obtained by solving the Plateau problem for its boundary, since the boundary of the solid torus is mean convex. The disk must be tangent to one of the Reeb leaves at some point, while lying locally to one side of this leaf. This violates the maximum principle for minimal surfaces [54],[36].

The absence of Reeb components is almost sufficient to allow the construction of a metric in which leaves are minimal. A closed curve in M that is never tangent to a leaf of F is called a *closed transversal.* The necessary condition was given by Sullivan [65]: every compact leaf of the foliation intersects some closed transversal. Such a foliation is called *taut.* See also Harvey and Lawson's work on calibrations [20].

Theorem 7.1. *Let M be a closed 3-manifold and let F be a codimension-one foliation of M. Then the following conditions are equivalent:*

1. *Every compact leaf of F intersects a closed transversal.*
2. *M admits a metric in which each leaf of F is a minimal surface.*

(3) *M admits a metric in which each leaf of F is area minimizing in its homology class.*

If the leaves are non-compact, area minimizing in a homology class means that each compact subsurface is minimizing in its homology class (rel boundary).

Sometimes such a taut metric is known, as with a product metric on a $\{\text{surface}\} \times S^1$. But often it is mysterious. An intriguing case is the hyperbolic one. It is known that there are many hyperbolic manifolds that fiber over S^1 with fiber a surface. Such a 3-manifold has a taut foliation consisting of all the fibers, and therefore has a metric in which each leaf is minimal.

Open Problem 13. *Is there a 3-manifold that fibers over the circle in which each fiber is a minimal surface in the hyperbolic metric?*

Thurston observed that the fibers of some hyperbolic bundles over the circle cannot all be minimal surfaces. Namely if there is a very short geodesic in the manifold, then a leaf going into the Margulis tube around this geodesic could not be minimizing in its homology class. A calculation shows that a smaller area homologous surface could be formed by cutting across a torus around the short geodesic, which looks something like a horotorus. However existence of a minimal foliation would force minimizing leaves into the Margulis tube.

An equivalent question asks whether there can be a family of stable minimal surfaces in a hyperbolic 3-manifold. If so, the family forms an analytic set [4], that closes up to create a bundle of minimal surfaces in some covering space.

8. The Spherical Space-Form Problem

The classification program for 3-manifolds starts by cutting a 3-manifold open along 2-spheres. The resulting prime pieces are uniquely determined [34],[42]. Understanding the pieces with finite fundamental group has two aspects

(1) What are the simply connected closed 3-manifolds?
(2) What are the manifolds which arise as their quotients?

The former problem is the Poincare Conjecture and the second is the Spherical Space-Form Problem. The Spherical Space-Form Problem itself has two parts

(1) What are the groups that can act on S^3 (or on a homotopy three sphere)?
(2) Is the action of a group on S^3 standard, i.e. conjugate to a fixed point free action of the rotation group $SO(4)$ on the round 3-sphere?

There are many restrictions known on the groups that can act freely on a homotopy 3-sphere, but a classification is still not achieved [33]. We will consider here the aspect of the problem that asks whether a subgroup of the rotation group $SO(4)$ that does act freely on S^3 always acts standardly. It suffices to consider cyclic subgroups to deal with most group actions, so we will restrict to this case. Livesay proved that $\mathbb{Z}_2$ acts standardly on S^3, and Rubinstein extended this to $\mathbb{Z}_{2^k}$ and $\mathbb{Z}_{3^k 2^l}$ for all $k, l \in \mathbb{N}$. An action of $\mathbb{Z}_5$, however, is still not known to be standard.

The problem admits an attack by minimal surface theory, as proposed by Pitts and Rubinstein. The Pitts-Rubinstein program involves finding an unknotted curve γ that is taken to itself by each element of the group. If such an invariant curve exists then a small neighborhood of the curve is an invariant unknotted solid torus, since the action is smooth. Since γ is unknotted, the complement of this solid torus is also a solid torus. Group actions on a solid torus are understood, via the Meeks-Yau Dehn's Lemma and the proof of the Smith Conjecture [38] [43], as we saw in Theorem 4.3. This leads to a proof that the action on the 3-sphere is standard.

Instead of searching for an invariant curve, we can look for an invariant solid torus, or for an invariant unknotted torus. Pitts and Rubinstein pointed out that if there is no invariant, unknotted torus, then one expects to find a sequence of unknotted minimal tori in the 3-sphere with unbounded index. Although this argument is not worked out completely, the idea is roughly as follows: By averaging a metric we can assume that the group acts as isometries. The space of unknotted tori $\mathcal{T}$ has known homotopy type, from the work of Hatcher on the Smale Conjecture [28]. If there is no fixed point of the group G acting on $\mathcal{T}$, then $\mathcal{T}/G$ has cohomology in infinitely many dimensions, and this in turn implies that a Morse function on $\mathcal{T}$ has critical points of unbounded index. The area function is defined on $\mathcal{T}/G$, and accordingly is expected to have critical points of unbounded index. More explicit arguments for why these higher index surfaces should exist can be made, though a rigorous proof is not yet available. If it were, the standardness of many group actions on the 3-sphere would be implied by a positive answer to the following question.

Open Problem 14. *Give a metric on the 3-sphere, is there an upper bound on the index of the unknotted, embedded, minimal tori in the 3-sphere?*

9. Normal surfaces

In the piecewise linear (PL) context, where one has a triangulated 3-manifold, an attempt to push the surface around until it becomes as simple as possible gives rise to what is called a *normal surface*. Normal surfaces are discrete analogs of minimal surfaces.

A triangulated 3-manifold is a decomposition of a 3-manifold into a union of tetrahedra, that intersect one another along lower dimensional simplices. We do not restrict to a combinatorial triangulation, so it's not forbidden for two tetrahedra to intersect along several faces or edges.

Definitions: *Normal triangles* are disks in a 3-simplex that meet three edges and three faces of the 3-simplex, and *normal quadrilaterals* are disks in a 3-simplex that meet four edges and four faces of the 3-simplex. An *elementary disk* is a normal triangle or quadrilateral. A *normal surface* in a triangulated 3-manifold is an embedded surface in M that intersects each 3-simplex in a disjoint union of elementary disks.

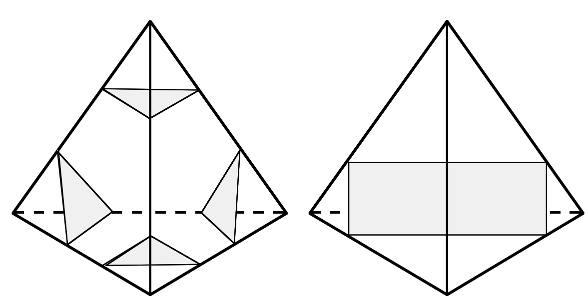

FIGURE 3. Pieces of a normal surface are triangles and quadrilaterals.

All four types of normal triangle can coexist disjointly in a 3-simplex. However as soon as one type of normal quadrilateral is around, the other two types of normal quadrilateral cannot be present, or else an intersection is forced.

A *compression* of a surface is an operation that squeezes away a non-trivial handle. Specifically, suppose that there is a disk whose interior misses the surface and whose boundary is a homotopically non-trivial curve on the surface. A thickening of this disk has boundary an annulus on the surface and two disks in the complement of the surface. A compression removes the annulus from the surface and replaces it with the two disks, resulting in a surface whose genus has dropped.

Any embedded surface can be compressed and isotoped to a union (possibly empty) of normal surfaces. To see this we introduce a measure of how complex a surface is relative to a given triangulation. The *weight* $w(F)$ of a surface F is the number of times it intersects the 1-skeleton of M. Weight gives an analog of area in the PL context. It gives the area in the limiting case when all area measure is concentrated near the 1-skeleton. Starting with an arbitrary surface, Kneser showed that one can carry out a series of compressions and weight decreasing isotopies until the surface becomes normal or empty.

Theorem 9.1. *Let F be an embedded surface in M. Then after a series of compressions, isotopies and removal of trivial 2-spheres, F becomes isotopic to a union (possibly empty) of disjoint normal surfaces.*

This result, implicit in the work of Kneser [34] and Haken[18], is the PL analog of Meeks-Simon-Yau's Theorem 5.3, except that in the Riemannian setting the metric may force the minimizer to be a double cover, while in the PL setting this type of collapse is unnecessary. Theorem 9.1 is proved by simplifying the intersections of a surface and the 1-skeleton, and compressing inside tetrahedra, until either the surface becomes normal or nothing is left [23].

Normal surfaces play a key role in the construction of algorithms to answer topological questions such as whether a knot is trivial and whether a 3-manifold is reducible. They are discrete objects described by a finite amount of data, the number of each type of triangle and quadrilateral that occurs in each tetrahedron. This data can be manipulated algebraically with the techniques of integer linear programming. In the 1990's this method was extended to solve an important problem, when Hyam Rubinstein discovered an algorithm to determine if a 3-manifold is homeomorphic to the 3-sphere [57]. A proof that this algorithm does indeed recognize the 3-sphere, based on thin position methods, was given by Abigail Thompson [66].

The idea behind the 3-sphere recognition algorithm is an insight from minimal surface theory. Suppose that M is a closed 3-manifold and we wish to determine whether it is a 3-sphere. We can assume that the homology of M is trivial, so that M is a homology sphere, as homology is straightforward to compute effectively. We then proceed as follows: Take a generic metric on M and pick a maximal collection of stable minimal 2-spheres in this metric. M is a 3-sphere if and only if each complementary piece is a ball, possibly with some further balls removed. If a complementary piece is a ball, then we can sweep it out by 2-spheres and find an unstable embedded 2-sphere, by a result of Simon and Smith, which in turn is based on earlier work of Pitts, Almgren and Simon [48],[2],[64]. See also Jost [32]. If the complementary piece is not homeomorphic to a ball, then it does not contain an unstable minimal 2-sphere. If it did we could push this unstable sphere to either side to get a slightly smaller 2-sphere. This smaller 2-sphere could then be deformed to a union of stable minimal 2-spheres or points by the Meeks-Simon-Yau Theorem. Any stable minimal 2-sphere lies on the boundary since our original collection was maximal. Thus this 2-sphere can be pushed out to the boundary on either side, and that implies the complementary piece is homeomorphic to a punctured 3-ball. Thus the problem of recognizing the 3-sphere could be solved if we knew all about the location of stable and unstable minimal 2-spheres in M.

In the Riemannian setting this is no easier, and maybe harder, than the original question. But in the PL setting, Rubinstein realized that stable and unstable minimal 2-spheres had PL analogs that could be found by a systematic procedure, leading to an algorithm. The analog of stable minimal surfaces were the normal surfaces. The analog of unstable minimal surfaces were modifications of normal surfaces introduced by Rubinstein, called *almost normal* surfaces. These look just like normal surfaces except in a single tetrahedron, where they either have an octagonal piece or a pair of normal disks connected by a tube. See Figure 4.

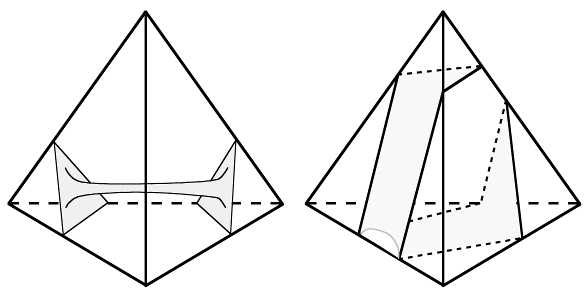

FIGURE 4. Pieces of an almost normal surface.

Schoen's curvature bound theorem states that a stable minimal surface in a Riemannian 3-manifold has principal curvatures that are bounded by a constant depending only on the geometry of the manifold [60]. It follows that an index-one minimal surface can have large principal curvature near one, but not two, points. Two disjoint neighborhoods, each containing a point of high principal curvature in a minimal surface, are each unstable, and lead to two independent regions of instability, implying that the index is at least two. The octagon or tube correspond in the PL context to one of these regions of instability.

The above discussion attempts to give the intuition behind almost normal surfaces. Once the analogies and definitions are established, one can work purely in the PL context and show that

(1) There is a maximal collection of disjoint normal 2-spheres in a 3-manifold.
(2) The complementary regions of this collection are punctured balls if and only if they contain an almost normal 2-sphere.
(3) The search for normal and almost normal spheres admits an algorithmic solution following the ideas of Haken [18].

M is homeomorphic to a 3-sphere if and only if each complementary region is a punctured ball, so the above combine to give an algorithm to recognize the 3-sphere. A closely related question is still open:

Open Problem 15. *Give an algorithm to determine if a 3-manifold is simply connected.*

10. Area and computational complexity

Recent work of Agol, Hass and Thurston [1] has shown an intriguing relation between the area of surfaces and the computational complexity of algorithms. The main result of [1] studies the complexity of algorithms to calculate the minimal genus of a spanning surface for a curve in a 3-manifold. The manifolds and curves are PL, allowing finite combinatorial descriptions. The 3-manifold is described as a collection of tetrahedra, with faces identified, and the curve as a collection of edges. Formally, the problem is described as a question with a yes or no answer.

Problem: 3-MANIFOLD KNOT GENUS
INSTANCE: A triangulated 3-dimensional manifold M, a knot K in the 1-skeleton of M,and a natural number g.
QUESTION: Does the knot K have genus $g(K) \leq g$?

In [1] it is shown that 3-MANIFOLD KNOT GENUS is **NP**-complete, and thus in the same complexity class as a large collection of extensively studied problems in other fields [13].

Area calculation problems seem at first to be less likely to fall into such classes, since area takes on real values that vary continuously as a surface is deformed. However we can recast the problem of determining minimal area into a form where its computational complexity can be measured. As with genus, we take a curve in a triangulated 3-manifold, but rather than ask whether it bounds a surface of genus bounded by k, we ask whether it bounds a surface of area less than k, where k is an integer. To describe a Riemannian metric on a 3-manifold with a finite amount of data requires limiting to a special class of metrics. We will use piecewise flat metrics, constructed from collections of tetrahedra and triangular prisms with flat metrics whose faces are identified by isometries. The curvature of such metrics is concentrated along their edges and vertices. A particular manifold in this class is described by a triangulation with a length assigned to each edge. The edge lengths of each tetrahedron and prism are rational numbers, and agree for identified edges. We do not require that the total angle around an edge is 2π, nor do we make any metric conditions at a vertex. Such piecewise flat metrics are described by a finite set of data, and can continuously approximate an arbitrary Riemannian metric. A curve is given as a collection of edges in the 1-skeleton of M. We refer to the following yes-no question as MINIMAL SPANNING AREA.

Problem: MINIMAL SPANNING AREA
INSTANCE: A 3-dimensional metrized PL manifold M, a 1-dimensional curve K in the 1-skeleton of M, and an integer k.
QUESTION: Does K bound a surface of area A satisfying $A \leq k$?

In analyzing its complexity, the size of an instance of this problem is taken to be the number of tetrahedra in the triangulation plus the log of the sum of the edge lengths plus $\log k$. We show that this problem is as hard as any **NP** problem.

Theorem 10.1. *MINIMAL SPANNING AREA is* **NP**-*hard.*

Sketch of Proof: We reduce in polynomial time an arbitrary instance of a known **NP**-complete problem to an instance of MINIMAL SPANNING AREA. We thus show that MINIMAL SPANNING AREA is at least as hard, up to polynomial time reduction, as this problem, which is called ONE-IN-THREE SAT.

Problem: ONE-IN-THREE SAT
INSTANCE: A set U of variables and a collection C of clauses over U

such that each clause $c \in C$ has $|c| = 3$.
QUESTION: Is there a truth assignment for U such that each clause in C has exactly one true literal?

Schaefer [59] established that ONE-IN-THREE SAT is **NP**-complete.

A general instance will have n variables and m clauses. As an example, consider the expression

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

with three variables and two clauses. A truth assignment setting x_2 to true and x_1, x_3 to false sets exactly one literal in each of the two clauses to be true, so ONE-IN-THREE SAT is satisfied.

We first set up a 2-dimensional version of MINIMAL SPANNING AREA. Given a boolean expression as above we construct a certain metrized 2-complex, a curve K in this 2-complex, and an integer k. The expression admits a satisfying assignment if and only if K bounds a surface of area less than k. For the expression above this metrized complex is shown in Figure 5.

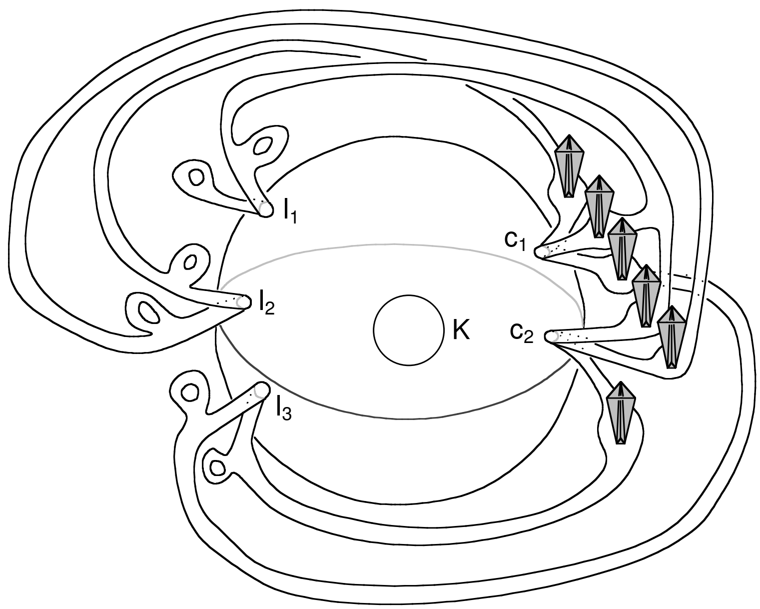

FIGURE 5. A branched surface with boundary curve K corresponding to the boolean expression $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$.

The surface branches off a punctured 2-sphere that has one boundary component called K, one additional boundary component on the left for each variable x_i, and one boundary component on the right for each clause. Each branching circle of the surface on the left corresponds to a variable x_i or $\bar{x}_i$, and a surface spanning K will go over one or the other of the two entering handles with odd algebraic degree. Each branching circle of the surface on the right corresponds to one of the clauses of the expression, and an odd number of the three entering handles is covered with odd degree by a spanning surface for K.

The metric on the branched surface is chosen so that each triangle in the triangulation is flat, and triangles that are not shaded have total area less than 1/2.

The protruding shaded prisms each have area close to one, and total area close to the number of clauses m.

A subsurface of the complex that is a spanning surface for K meets each boundary component of the punctured 2-sphere an odd number of times. It thus goes over at least one shaded prism disk of the three associated to each clause curve c_i, and has area at least m, where m is the number of clauses (two in our example). Any spanning surface meeting more than one shaded prism disk associated to a single clause curve c_i must have area greater than $m+1$. Furthermore, a satisfying assignment for ONE-IN-THREE SAT leads to an embedded spanning surface with the satisfying values of the variables selecting the branches of the surface. It therefore picks out a spanning surface having area less than $m+1$. So such a surface exists if and only if the expression has a satisfying assignment. Thus an instance of ONE-IN-THREE SAT can be reduced to an instance of MINIMAL SPANNING AREA in polynomial time, and MINIMAL SPANNING AREA is **NP**-hard.

There is one final point to address. We have worked with a branched surface rather than a 3-manifold. To pass to a 3-manifold we thicken each triangle in the branched surface to a triangular prism with the product metric. In this metric projection is area non-increasing and a least area surface spanning K must lie on the branched surface. Ppassage to a closed manifold is obtained by a doubling construction. See [1] for details. □

References

[1] I. Agol, J. Hass and W. P. Thurston, The computational complexity of Knot Genus and Spanning Area, preprint, (2001).

[2] F. J. Almgren Jr. and L. Simon, Existence of embedded solutions of Plateau's Problem, Annali Scuola Normale Superiore Pisa, Serie IV, Vol. VI (1979), 447-495.

[3] H.W. Alt, Verzweigungspunkte von H-Flachen. I. (German) Math. Z. 127 (1972), 333–362.

[4] R.Bohme and F. Tomi, Zur Struktur der Lsungsmenge des Plateauproblems (German) Math. Z. 133 (1973), 1–29.

[5] F. Bonahon and J.P. Otal, Scindements de Heegaard des espaces lenticulaires, Ann. Sci. Ecole Norm. Sup. (4) 16 (1983), no. 3, 451–466 (1984).

[6] R. Courant, Dirichlet's principle, conformal mapping, and minimal surfaces. With an appendix by M. Schiffer. Reprint of the 1950 original. Springer-Verlag, New York-Heidelberg, 1977.

[7] M. Dehn, Uber die Topologie der dreidimensionalen Ruames Math. Ann. 69 (1920) 137-168.

[8] J. Douglas, Solution of the problem of Plateau, Trans. A.M.S. 33 (1931), 263-321.

[9] J. Eells and L. Lemaire, Another report on harmonic maps. Bull. London Math. Soc. 20 (1988), no. 5, 385–524.

[10] W.H. Fleming, On the oriented Plateau problem. Rend. Circ. Mat. Palermo (2) 11 1962 69–90.

[11] H. Federer, and W.H. Fleming. Normal and integral currents. Ann. of Math. (2) 72 1960 458–520.

[12] M. Freedman, J. Hass and P. Scott, Least area incompressible surfaces in 3-manifolds, Invent Math. 7 (1983), 609-642.

[13] M. Garey and D.S. Johnson, Computers and intractability. A guide to the theory of NP-completeness, W. H. Freeman and Co., San Francisco, 1979.

[14] R. Gulliver, Regularity of minimizing surfaces of prescribed mean curvature, Ann. of Math. (2) 97 (1973), 275–305.

[15] R. Gulliver, A minimal surface with an atypical boundary branch point. Differential geometry, 211–228, Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991.

[16] R. Gulliver and F. Leslie, On boundary branch points of minimizing surfaces, Arch. Rat. Mech. Anal. 52 (1973), 20-25.

[17] R. Gulliver and J. Spruck, On embedded minimal surfaces, Ann. of Math. (2) 103 (1976), no. 2, 331–347.
[18] W. Haken, Theorie der Normalflächen: Ein Isotopiekriterium für den Kreisknoten, Acta Math., 105 (1961) 245–375.
[19] R. Hardt and L. Simon, Boundary regularity and embedded solutions for the oriented Plateau problem. Ann. of Math. (2) 110 (1979),
[20] R. Harvey and H.B. Lawson, Calibrated geometries, Acta Math. 148 (1982), 47–157.
[21] J. Hass, Minimal surfaces in foliated manifolds, Comment. Math. Helv. 61 (1986), no. 1, 1–32.
[22] J. Hass, Singular curves and the Plateau problem, International J. of Math. 2, (1991) 1-16.
[23] J. Hass, Algorithms for recognizing knots and 3-manifolds, Chaos, Solitons and Fractals, 9 (1998) 569–581.
[24] J. Hass, J. C. Lagarias and N. Pippenger, The computational complexity of Knot and Link problems, Journal of the ACM, 46 (1999) 185–211.
[25] J. Hass and J. H. Rubinstein, One-sided closed geodesics on surfaces, Mich. Math. J. 33 (1986) 155-168.
[26] J. Hass, J. H. Rubinstein and P. Scott, Compactifying coverings of 3-manifolds, J. Differential Geometry 30, (1989) 817-832.
[27] J. Hass and P. Scott, The existence of least area surfaces in 3-manifolds, Trans. Amer. Math. Soc. 310 (1988), 87-114.
[28] A. Hatcher, A proof of a Smale conjecture, $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$. Ann. of Math. (2) 117 (1983), no. 3, 553–607.
[29] J. Hempel, 3-manifolds, Annals of Math. Studies, 86, Princeton University Press 1976.
[30] W. Jaco, Lectures on the topology of 3-manifolds, Conf Board of Math Sci, 43 Amer. Math. Soc. Providence R.I. 1977.
[31] W. Jaco and J. H. Rubinstein, PL Equivariant Surgery and Invariant Decompositions of 3-Manifolds, Advances in Math., 73 (1989) 149–191.
[32] J. Jost, Embedded minimal surfaces in manifolds diffeomorphic to the three-dimensional ball or sphere. J. Differential Geom. 30 (1989), no. 2, 555–577.
[33] R. Kirby, Problems in low-dimensional topology, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI, 1997.
[34] H. Kneser, Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten, Jahresbericht Math. Verein., 28 (1929) 248–260.
[35] H. B. Lawson, Lectures on minimal submanifolds, 9 Publish or Perish, Berkeley 1980.
[36] W. Meeks, Lectures on Plateau's problem, IMPA, Rio de Janeira, 1978.
[37] W. Meeks and G.P. Scott. Finite group actions on 3-manifolds. Invent. Math. 86 (1986), no. 2, 287–346.
[38] W. Meeks and S.T. Yau, Group actions on $\mathbb{R}^3$. The Smith conjecture (New York, 1979), 167–179,
[39] W. Meeks and S.T. Yau, Topology of three-dimensional manifolds and the embedding problems in minimal surface theory. Ann. of Math. (2) 112 (1980), no. 3, 441–484.
[40] W. Meeks and S.T. Yau, The classical Plateau problem and the topology of three-dimensional manifolds, The embedding of the solution given by Douglas-Morrey and an analytic proof of Dehn's lemma, Topology 21 (1982), no. 4, 409–442.
[41] W. Meeks, L. Simon and S.T. Yau, Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. Ann. of Math. (2) 116 (1982), no. 3, 621–659.
[42] J. Milnor, Morse theory, Annals of Math. Studies, 51 Princeton University Press 1963.
[43] Morgan, John W. The Smith conjecture (New York, 1979), 3–6, Pure Appl. Math., 112, Academic Press, Orlando, FL, 1984.
[44] C.B. Morrey, The problem of Plateau on a Riemannian Manifold, Annals of Math. 49 (1948) 807-851.
[45] R. Osserman, A survey of minimal surfaces. Second edition. Dover Publications, Inc., New York, 1986.
[46] R. Osserman, A proof of the regularity everywhere of the classical solution to Plateau' s problem, Annals of Math. 91 (1970), 550-569.
[47] C.D. Papakyriakopolous, Dehn's lemma and the asphericity of knots, Annals of Math. 66 (1957), 1-26.
[48] J. Pitts, Existence and regularity of minimal surfaces on Riemannian manifolds, Lecture Notes, Princeton University Press, Princeton, 1981.

[49] J. Pitts and J.H. Rubinstein, Existence of minimal surfaces of bounded topological type in 3-manifolds, Proc. Centre Math. Applic., Australian National University 10, (1985), 163-176.

[50] J. Pitts and J.H. Rubinstein, Applications of minimax to minimal surfaces and the topology of 3-manifolds, Proc. Centre Math. Applic., Australian National University 12, (1987), 137-170.

[51] J. Pitts and J.H. Rubinstein, The topology of minimal surfaces in Seifert fibered spaces, Mich. Math. Journal 42 (1995), 525-535.

[52] J. Pitts and J.H. Rubinstein, Equivariant minimax and minimal surfaces in geometric 3-manifolds, Bulletin Amer. Math. Soc. 19 (1988), 303-309.

[53] J. Plateau, Statique experimentale et theorique des liquides soumis auz seules forces moleculaires, Gathier-Villars, Paris, 1873.

[54] M.H. Protter and H.F. Weinberger, Maximum principles in differential equations. Prentice-Hall, Inc., Englewood Cliffs, N.J. 1967

[55] T. Rado, The problem of least area and the problem of Plateau, Math. Z. 32 (1930), 763-796.

[56] T. Rado, On the Problem of Plateau. Chelsea Publishing Co., New York, N. Y., 1951.

[57] J. Rubinstein, An algorithm to recognize the 3-sphere, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zurich, 1994), Birkhauser, Basel, 601–611.

[58] J. Rubinstein, These proceedings.

[59] T.J. Schaefer, The complexity of satisfiability problems, Proc 10th Ann. ACM. Symp. on Theory of Computing, ACM, NY, (1978) 216-226.

[60] R. Schoen, Curvature estimates for stable minimal surfaces in three-dimensional manifolds, Seminar on minimal submanifolds, Ann of Math. Studies 103, Princeton University Press 1983. 111-126.

[61] R. Schoen and S.T. Yau, Existence of incompressible surfaces and the topology of 3-dimensional manifolds with non-negative scalar curvature, Annals of Math. 119 (1979), 127-142.

[62] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983), 401-487.

[63] P. Scott, There are no fake Seifert fibre spaces with infinite π_1, Ann. of Math. (2) 117 (1983), no. 1, 35–70.

[64] F. Smith, On the existence of embedded minimal 2-spheres in the 3-sphere, endowed with an arbitrary Riemannian metric, Ph.D. thesis, supervisor L. Simon, University of Melbourne 1982.

[65] D. Sullivan, A homological characterization of foliations consisting of minimal surfaces, Comment. Math. Helv. 54 (1979), no. 2, 218–223.

[66] A. Thompson, Thin position and the recognition problem for S^3, Math. Res. Lett. 1 (1994), 613–630.

[67] F. Tomi and A. Tromba, Extreme curves bound embedded minimal surfaces of the type of the disc. Math. Z. 158 (1978), no. 2, 137–145.

[68] W. Thurston, Three-dimensional geometry and topology, Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

[69] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. (2) 87 1968 56–88.

[70] B. White, The space of minimal submanifolds for varying Riemannian metrics, Indiana Math. Journal 40 (1991), 161-200.

[71] S. T. Yau, Open problems in geometry, Proc of Sym in Pure and Appl Math., 54, I Amer. Math. Soc. Providence R. I. 1993.

Department of Mathematics, University of California, Davis, California 95616
E-mail address: `hass@math.ucdavis.edu`